Legendrian curves on fiber surfaces

山本 亮介 (Ryosuke Yamamoto)
大阪大学大学院理学研究科 研究生

2004.1.31

Abstract

I will talk a way of constructing a (positive) contact structure of closed orientable 3-manifold M from an open-book decomposition of M, and we will see a condition that a simple closed curve on a fiber surface becomes a Legendrian curve in the contact structure.

1 Preliminary

Let M be a closed smooth orientable 3-manifold, K a fibered knot in M with a fiber surface F. Let $E(K)$ denotes an exterior of K in M.

Definition 1.1. An *open-book decomposition* (F, φ) of M consists of the knot K, called the *binding*, and a fibration $\varphi : E(K) \to S^1$. A fiber surface F is called a *page*. Note that

$$E(K) \cong F \times [0,1]/(x,1) \sim (h(x),0)$$

, where $h : F \to F$ is a homeomorphism fixing ∂F pointwize.

Definition 1.2 ([1]). A *contact form* on M is a smooth 1-form ω such that

$$\omega \wedge d\omega \neq 0$$

at each point. A *contact structure* (M, ξ) is a 2-plane field $\xi = \ker \omega$ on M. We call a contact structure is *positive* when $\omega \wedge d\omega > 0$.

Example 1.3. Let (R^3, ξ_0) be a contact structure on R^3 defined by the contact form $\omega_0 = xdy - ydx + dz$.

$$(xdy - ydx + dz) \wedge d(xdy - ydx + dz) = 2dz \wedge dx \wedge dy > 0$$

We call this structure the *standard* contact structure on R^3 (see Figure A).

Definition 1.4. A contact structure on M is *supported* by an open-book decomposition (F, φ) if it is defined by a 1-form ω such that
(1) on each fiber F, $d\omega|_F > 0$,
(2) ω is transverse to K and orients K as the boundary of $(F, d\omega)$.

Definition 1.5. A simple closed curve γ is called *Legendrian* if for every $x \in \gamma$, $T_x\gamma \subset \xi_x$ (i.e., γ is always tangent to ξ).

2 Construction of a contact structure

The aim of this section is to construct a positive contact form ω on M supported by (F, φ) from the standard contact strucure on S^3, through a *simple cover* $p : M \to S^3$. This construction is based on [3].

- Put $D = \{(x, y) \in R^2 | x^2 + y^2 \leq 1\}$, and 1-form $\alpha_0 = xdy - ydx$ on D. Note that α_0 has the propaties follows:
  (1) $d\alpha_0 = 2dx \wedge dy$ is a volume form on D,
  (2) α_0 orients ∂D as the boundary of $(D, d\alpha_0)$.

Definition 2.1. Let S be a orientable surface with boundary and D a 2-disk. A branched covering map $p : S \to D$ is called a *simple cover* with d sheets if there is a finite set $Q \subset \operatorname{Int} D$ and each $x \in D$ has a disk neighbourhood U as follows:
(1) if $x \notin Q$ then $p|_{p^{-1}(U)}$ is a trivial d-sheeted cover,
(2) if $x \in Q$ then $p^{-1}(U)$ has $d - 1$ components, one of which is a disk projecting to U as double cover branched over x, and the others are disks projecting homeomorphically.

- Let $p : F \to D$ be a simple cover, and set $\alpha = p^*\alpha_0$. We have that (1) $d\alpha$ is a volume form on F and (2) α orients ∂F as the boundary of $(F, d\alpha)$.

- The 1-form β on $E(K) = F \times [0,1]/(x,1) \sim (h(x),0)$ such that

$$\beta|_{F\times t} = (1-t)\alpha + th^*\alpha, \qquad t \in [0,1]$$

has the properties (1) and (2) in Definition 1.4, and may not be contact. Let $ds = \varphi^* d\theta$, where $d\theta$ is a volume form on S^1. For a sufficient large constant N, the form

$$\omega = \beta + Nds$$

is a contact form on $E(K)$. We can extend ω to M smoothly, and then ω is a contact form on M supported by (F, φ).

- By [2], there is the homeomorphism $b : D \to D$ such that

$$b \circ p = p \circ h.$$

In a similar way, we can construct a contact form ω_0 on S^3 suppored by the (trivial) open-book decomposition of S^3 with fibers $D \times t$ and the monodromy map b.

Proposition 2.2. *Let (M, ξ) be a contact structure on M supported by (F, φ). Then there is a closed braid $\widehat{b} \subset S^3$ with axis L, and a simple cover $p : M \to S^3$ branched over $\widehat{b}$ such that $F \times t = p^{-1}(D \times t)$ for each $t \in [0,1]$. A contact form $\omega = p^*\omega_0$ defines a contact structure ξ' on M which is isotopic to ξ.*

3 Results

Theorem 3.1. *Let (M, ξ) be a contact structure supported by an open-book decomposition (F, φ) of M, and c a simple closed curve on F. There is a positive contact structure ξ' on M isotopic to ξ such that c is a Legendrian curve in (M, ξ') if and only if c is not null-homologous.*

Corollary 3.2. *Let K be a fibered knot in S^3 with a fiber surface F and a fibration φ. If there is a non-seperating loop c on F such that a tublar neighbourhood of c in F is an unknoted, untwisted annulus in S^3, then $\xi_{(F,\varphi)}$ is overtwisted.*

References

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