A surgery description of homology solid tori and its applications to the Casson invariant

Yukihiro TSUTSUMI*
Keio University (JSPS Research Fellow)

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1 Notation

M, H : an oriented 3-manifold, an oriented $\mathbb{Z}HS^3$,

$K \subset H$: a knot in H,

$\mathcal{L}$ (resp. $\mathcal{K} = (K, \gamma)$): a rational framed link (resp. knot) in H,

$X' = \chi(X; \mathcal{L})$: the object obtained from X by surgery along (K, γ), and we call $(X; \mathcal{L})$ a *surgery description for* X',

lk : the linking number,

λ: the Casson invariant,

$\Delta_{K\subset H}(t)$: the symmetric Alexander polynomial of K in H,

$\varphi : M_K^r \to H$: the r-fold cyclic cover of H branched along a knot $K \subset H$.

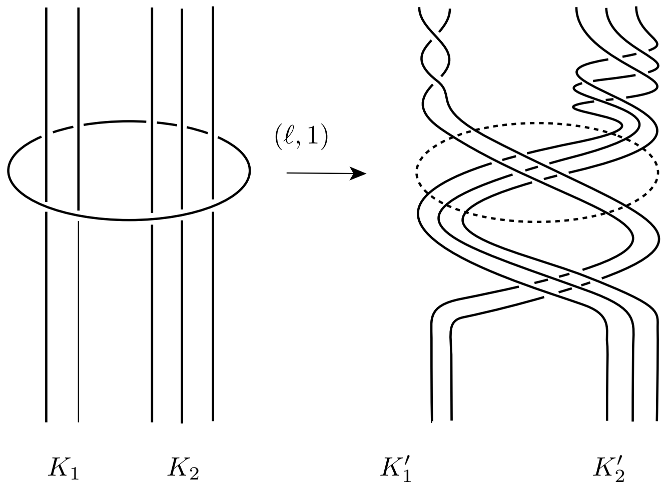
Lemma 1.1. *Let $K_1 \cup K_2$ be a link in an $\mathbb{Z}HS^3$ H. Let $(\ell, 1/n)$ be a $1/n$-framed knot in H disjoint from $K_1 \cup K_2$. Then in $H' = \chi(H; (\ell, 1/n))$,*

$$\mathrm{lk}_{H'}(K_1', K_2') = \mathrm{lk}_H(K_1, K_2) - n \cdot \mathrm{lk}_H(K_1, \ell) \cdot \mathrm{lk}_H(K_2, \ell),$$

where $K_i' = \chi(K_i; (\ell, 1/n))$.

*email: yukihiro@math.keio.ac.jp, Department of Mathematics, Faculty of Science and Technology, Keio University, Hiyoshi 3-14-1, Kohoku-ku Yokohama 223-8522, Japan.

†See [20] for the details.



In particular, when $\mathrm{lk}(K_i, \ell) = 0$*, we have*

$$\mathrm{lk}_{H'}(\chi(K_1; (\ell, 1/n)), \chi(K_2; (\ell, 1/n))) = \mathrm{lk}_H(K_1, K_2).$$

Definition 1.2. A link $L = K_1 \cup K_2 \cup \cdots \cup K_m$ is called a *boundary link* if K_i bound mutually disjoint compact, connected, oriented surfaces S_i.

For a boundary link, we have $\mathrm{lk}(K_i, K_j) = 0$. Furthermore, by using Lemma 1.1 we have that $\Delta_{K_j \subset H}(t) = \Delta_{\chi(K_j;(K_i,\varepsilon_i)) \subset \chi(H;(K_i,\varepsilon_i))}(t)$. The framing number for K_i is unchanged by surgery along other components.

2 Boundary links and the Casson invariant

In 1985, A. Casson[1] introduced an $\mathbb{Z}$-valued invariant λ for oriented $\mathbb{Z}\mathrm{H}S^3$s which satisfies the following properties:

- $\lambda(S^3) = 0$ ($\lambda(H) = 0$ if $\pi_1(H) = 1$)
- $\lambda(-H) = -\lambda(H)$
- $\lambda(H_1 \# H_2) = \lambda(H_1) + \lambda(H_2)$
- $\lambda(\chi(H; (K, 1/n))) - \lambda(H) = \frac{n}{2}\Delta''_{K \subset H}(1)$ [2] [3]

[1] See [1], [16]

[2] This is called the Casson surgery formula. See [6] for the link version.

[3] This is the only restriction in some sense. That is, given two knot-Alexander polynomials $\Lambda_1(t)$ and $\Lambda_2(t)$ with $\varepsilon_1\Lambda''(1) = \varepsilon_2\Lambda''(1)$, there are two knots K_1, K_2 in S^3 with $\Delta_{K_i}(t) = \Lambda_i(t)$ such that $\chi(S^3; (K_1, \varepsilon_1)) = \chi(S^3; (K_2, \varepsilon_2))$ [19, Theorem 1.1]. Can you construct three more knots?

Note that $\frac{1}{2}\Delta''_K(1) = a_2(K)$, where a_n denotes the nth coefficient of the Conway polynomial $\nabla_K(z) = \Delta_K(t)|_{t^{-1/2}-t^{1/2}=z}$.

Let H be an $\mathbb{Z}HS^3$, and

$$H = H_1 \overset{(K_1,\varepsilon_1)}{\longrightarrow} H_2 \overset{(K_2,\varepsilon_2)}{\longrightarrow} \cdots \overset{(K_{n-1},\varepsilon_{n-1})}{\longrightarrow} H_n \overset{(K_n,\varepsilon_n)}{\longrightarrow} H'$$

a sequence of surgeries, where K_i is a knot in H_i and $\varepsilon_i \in \{-1, +1\}$. Then there is a boundary link[4] $L = K'_1 \cup \cdots \cup K'_n$ in H such that

$$\chi(H; (K'_1, \varepsilon_1), (K'_2, \varepsilon_2), \cdots, (K'_n, \varepsilon_n)) = H'$$

Then by the Casson surgery formula, we have that

$$\lambda(H') - \lambda(H) = \sum_{i=1}^{n} \varepsilon_i a_2(K'_i).$$

3 Surgery descriptions of knots and cyclic branched covers

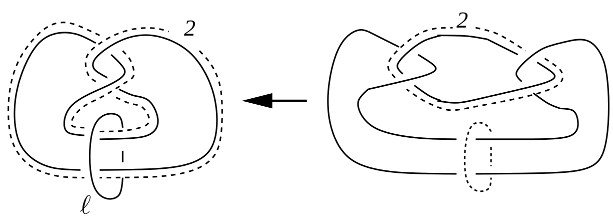
Let $\varphi : M^r_K \to H$ denote the r-fold cyclic covering branched along K. We can compute $|H_1(M^r_K; \mathbb{Z})|$ from the Alexander matrix. Hoste showed the following:

Proposition 3.1 ([6, Theorem 3.2]). *Let $\mathcal{D}K$ be the untwisted doubled knot about K in H. Then $\lambda(M^r_{\mathcal{D}K}) = \lambda(H) + 2a_2(K)$.*

Recall that $\Delta_{\mathcal{D}K}(t) = 1$. Thus, we see that $\lambda(M^r_K)$ cannot be computed from $\Delta_K(t)$ in general.

Let $K \cup \ell$ be a link in H with $\mathrm{lk}(K, \ell) = 0$. Let K^+ denote the preferred longitude for K. Namely $\mathrm{lk}(K, K^+) = 0$. Then in M^r_ℓ, $\varphi^{-1}(K)$ consists of r components. Let $\bar{K}$ be a component of $\varphi^{-1}(K)$ which forms a knot in M^r_ℓ. Let $\bar{K^+}$ be the component of $\varphi^{-1}(K^+)$ corresponding to $\bar{K}$.

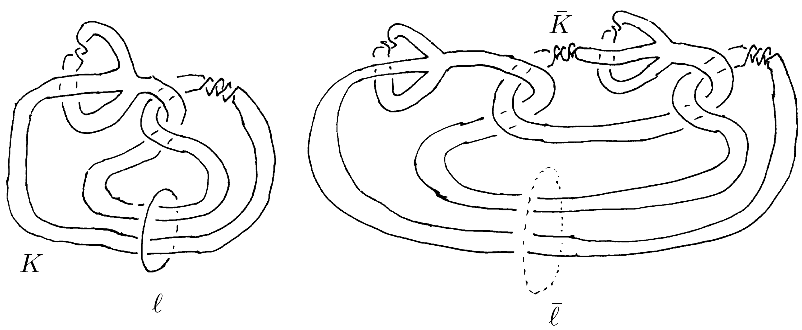
Put $\alpha_r(K, \ell) = \mathrm{lk}(\bar{K}, \bar{K^+})$. This is an even integer[5]. If $r' >$ (the wrapping number), then $\alpha_{r'}(K, \ell) =$ (const).



For this Whitehead link $K \cup \ell$, we have $\alpha_r(K, \ell) = 2$ $(r > 1)$. In general, $\Delta_{\bar{K}}(t) \neq \Delta_K(t)$ (even if $\alpha_r(K, \ell) = 0$, even if $K \cup \ell$ is a boundary link.)

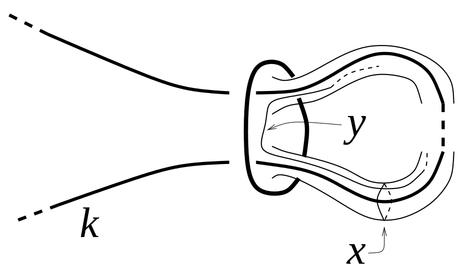
[4]In [11], C. Lescop showed that when $\lambda(H') = \lambda(H)$ there is such a boundary link in H with $\Delta_{K'_i}(t) = 1$. See [8], [19] for more generalization of this result.

[5]Exercise: Relate $\alpha_r(K, \ell)$ to the Alexander matrix of the knot $\chi(K, (\ell, \pm 1))$.



$\Delta_K(t) = 1$ (K is a trivial knot) and $\Delta_{\bar{K}}(t) = 1 + 2(t^{1/2} - t^{-1/2})^2$. [6]

Theorem 3.2. *Any knot K in an integral homology sphere H has a surgery description $(k; (k_i, \varepsilon_i)_{i=1,2,\ldots,n})$, where k is a trivial knot in S^3 and k_i's are mutually disjoint knots in $E(k)$, with the following properties:*



- $\varepsilon_i \in \{-1, +1\}$.
- $k_1, \ldots k_N$ ($N \leq n$ ($N = n$ when $H = S^3$)) bound mutually disjoint genus-one Seifert surfaces S_i's in $E(k)$ on which there are non-separating simple closed curves x_i and y_i such that:
  - x_i intersects y_i transversely in a single point,
  - x_i bounds a disk D_i with $D_i \cap S_i = \partial D_i = x_i$ which meets k in a single point,
  - $\mathrm{lk}(y_i, k) = 0$.
- Each k_j ($N < j \leq n$ if $N \neq n$) bounds a Seifert surface S_j disjoint from k such that
  - $\mathrm{lk}(k, \ell) = 0$ for each simple closed curve ℓ on S_j,
  - S_1, S_2, $\ldots$, S_n are mutually disjoint,
- $a_2(K) = -\sum_{i=1}^{N} \varepsilon_i \mathrm{lk}(y_i, y_i^+)$, where y_i^+ denotes the push-forward of y_i with respect to S_i.

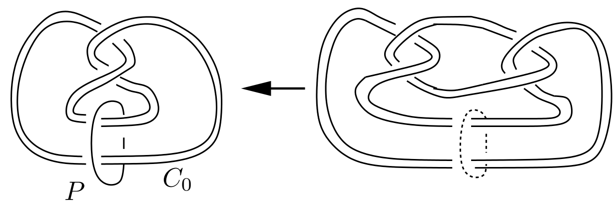
[6] This contradicts the "statement" of [2, Proposition A.7].

- $\lambda(H) = \sum_{i=N+1}^{n} \varepsilon_i a_2(k_i)$.

To illustrate usefulness of such a surgery description, we show the following:

Theorem 3.3. *Let K be a satellite knot about a companion C, with the pattern (P, C_0). When the winding number is zero,*

$$\lambda(M_K^r) = \lambda(M_P^r) - r\alpha_r(P, C_0)a_2(C).$$



For the surgery description $(k; (k_i, \varepsilon_i)_{i=1,2,\ldots,n})$, in M_P^r the r copies $(\bar{k}_i, \varepsilon_i)_i$ of $(k_i, \varepsilon_i)_i$ give a sequence $M_P^r \to \cdots \to M_K^r$. It is easy to see that $\mathrm{lk}(\bar{x}_i, x^{\bar{+}}{}_i) = \alpha_r(P, C_0)$ and $\Delta_{\bar{k}_i}(t) = 1 + \alpha_r(P, C_0)\mathrm{lk}(y_i, y_i^+)(t^{1/2} - t^{-1/2})^2$.

4 Realization problem

Exercise 4.1. For any integer λ, there is a knot K in S^3 such that $\Delta_K(t) = 1$ and $\lambda(M_K^r) = 2r\lambda$ for any natural number $r > 1$. (Hint: Use Hoste's theorem (Proposition 3.1).)

As an application of Theorem 3.3, we have the following:

Proposition 4.2. *Let r_0 be a natural number, and λ an integer. Then there is a knot K in S^3 such that $\Delta_K(t) = 1$, $\lambda(M_K^r) = 2r\lambda$ for any natural number $r \neq 1$, r_0, and $\lambda(M_K^{r_0}) = 0$.*

Problem 4.3. *Let K be a knot in S^3 with $\Delta_K(t) = 1$ (or $a_2(K) = 0$.) Then $\lambda(M_K^r)$ is divided by $2r$. (For unknotting number one knots with $\Delta_K(t) = 1$, this problem is true.)*

Exercise 4.4. Let K be a knot in S^3. Then $H_1(M_K^2; \mathbb{Z})$ has no elements of order two. Namely, the order of $\mathrm{Tor}(H_1(M_K^2; \mathbb{Z}))$ is odd.

Exercise 4.5. Given a finitely generated abelian group G without elements of order two, there are knots K in S^3 with $H_1(M_K^2; \mathbb{Z}) = G$.

Exercise 4.6. Study the above propositions for 3-fold cyclic branched covers. ($H_1(M_K^3; \mathbb{Z}) = G + G$, $|\mathrm{Tor}(G)|$ is odd.)

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