ON THE KHOVANOV INVARIANT FOR LINKS

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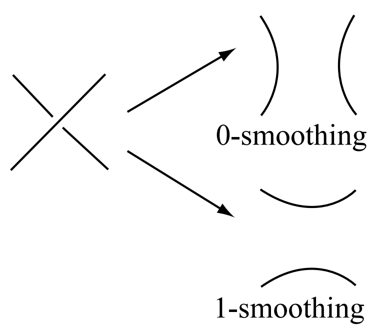
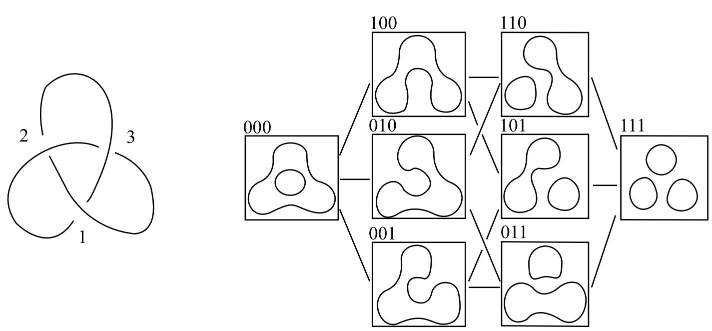
Abstract. In 1999, M. Khovanov constructed an invariant of oriented links. It is a family of (co)homology groups whose graded Euler characteristic is the Jones polynomial. Though his definition is algebraically very complicated, O. Viro simplified it in 2002. In this talk, I will introduce the Khovanov invariant defined by O. Viro and report its related topics.

1. The Khovanov invariant defined by O. Viro

Definition 1. Let D be a diagram of a link in S^3. We exchange a neighborhood of each crossing point of D for either of the following two pictures on the right side. It is called a *Kauffman state* or a *state* of D for short that the disjoint circle(s) given by exchanges like this.



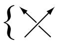
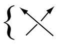
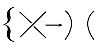
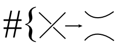
Example 1. The states of the left diagram are like these.



Date: 1st Feburuary.

Definition 2. Let s be a state of D. If we assign a plus or minus sign to each circle of s, it is called an *enhanced Kauffman state* or an *enhanced state* of D or s for short.

From now on, we assume that links are oriented and crossing points of each diagram are enumerated by natural numbers $1, 2, \cdots, n$.

Definition 3. Define $w(D) := \#\{$  $\} - \#\{$  $\}$ and $\sigma(s) := \#\{$  $\} - \#\{$  $\}$. Let $\tau(S)$ be the summation of the signs given to the circles of an enhanced state S. Then,

$$i(S) := \frac{w(D) - \sigma(s)}{2} \quad j(S) := -\frac{\sigma(s) + 2\tau(S) - 3w(D)}{2}$$

Khovanov chain complices

Definition 4. We call the following three free abelian group a *Khovanov chain group* respectively.

$C(D) := \left\{ \sum_l a_l S_l | a_l \in \mathbb{Z}, S_l : \text{an enhanced state of } D \right\}$,

$C^i(D) := \left\{ G \in C(D) | G = \sum_l a_l S_l, i(S_l) = i \right\}$,

$C^{i,j}(D) := \left\{ G \in C^i(D) | G = \sum_l a_l S_l, j(S_l) = j \right\}$.

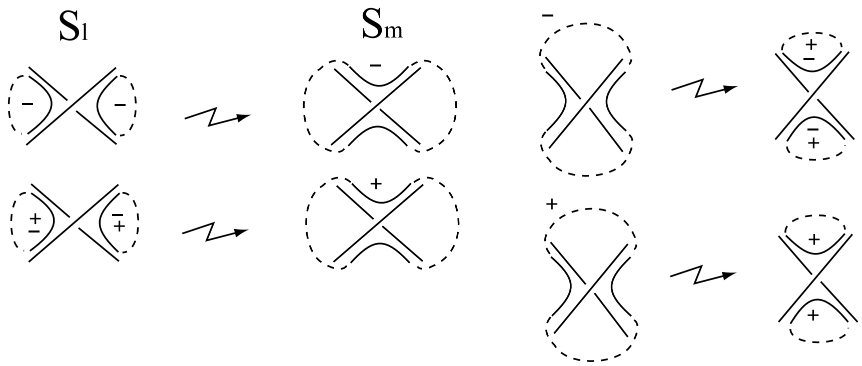
Definition 5. A homomorphism $\partial : C(D) \longrightarrow C(D)$ is defined as below by the incidence numuber (S_l, S_m). $(S_l, S_m \in C(D).)$

$$\partial(\sum_l a_l S_l) = \sum_{l,m} a_l (S_l : S_m) S_m$$

$$(S_l : S_m) = \begin{cases} (-1)^t & \big((S_l, S_m) \text{ satisfys Condition } 1, 2\big) \\ 0 & \big(\text{otherwise}\big) \end{cases}$$

Condition 1. Only at one crossing point of D (Let it have number k), the smoothings of S_l and S_m differ and at this crossing point the smoothing of S_l is 0, while the smoothing of S_m is 1.

Condition 2. The common circles of S_l and S_m have the same signs, and on the signs of the circles of S_l, S_m adjacent to the kth crossing point, they satisfy one of the situations in the next page.



t is the number of 1-smoothings in S_l numerated with numbers greater than k.

Lemma. $\partial^{i+1,j} \circ \partial^{i,j} = 0$.

Notation. $\mathcal{H}^{i,j}(D) := Ker\partial^{i,j}/Im\partial^{i-1,j}$

Definition 6. Let L be an oriented link in S^3 and let D be a diagram of L.
(1) The isomorphism class of $\mathcal{H}^{i,j}(D)$ is called the *Khovanov homology* of L and is denoted by $\mathcal{H}^{i,j}(L)$.
(2)$Kh(L)(t,q) := \sum_{i,j\in\mathbb{Z}} t^i q^j \operatorname{rank} \mathcal{H}^{i,j}(D)$. The polynomial is called the *Khovanov polynomial* of L.

Theorem. $Kh(L)(-1,q) := \hat{J}(L)(q)$.
($\hat{J}(L)(q)$ *is a version of the Jones polynomial such that* $< \emptyset >= 1$.)

2. Related topics

· **A categorification of the Kauffman bracket skein module**

F を向き付けられた曲面、I を単位区間とする。M. M. Asaeda、J. H. Przytycki、A. S. Sikora 3 氏の共同研究で、O. Viro による Khovanov homology の定義を基に、少なくとも $F \times I$ に関する Kauffman bracket skein module の categorification に成功したようである。

· **The Khovanov polynomials for the links with trivial Jones polynomials**

絡み目の Jones 多項式が trivial であるとは、自明な絡み目の Jones 多項式と一致することであるとする。M. Thistlethwaite が、2001 年に Jones 多項式が trivial になるような自明でない絡み目の例を 3 つ発表した。それらの Khovanov 多項式を

A. Shumakovitch が作成した KhoHo を用いて計算することが出来た。3 つとも、互いに異なる non-trivial な多項式になる。

・ A categorification of the HOMFLY polynomial

P. Ozsváth、Z. Szabó が構成した knot Floer homology は、構成方法は全く異なるのだが Khovanov homology と非常に似た性質を持っている。例えば、あるオイラー標数を取ると Khovanov homology では Jones 多項式と一致するのだが、knot Floer homology では Alexander 多項式と一致する。また交代結び目に関して
Khovanov homology(のランク) は Jones 多項式と結び目の符号数で決定できるのに対し、knot Floer homology は Alexander 多項式と結び目の符号数で決定できる。このため、P. Ozsváth、Z. Szabó の論文において「共通の一般化を考えるのは自然なことである。」という指摘がなされていた。

そして、最近 M. Khovanov と L. Rozansky による共著で、それに非常に近いものを構成したという preprint が発表された。具体的には、n が 2 の時に Jones 多項式、n が 0 の時に Alexander 多項式となるように specialize した 1 変数の HOMFLY 多項式に関する categorification を行っている。

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