BRAID INDICES OF SURFACE-KNOTS AND COLORINGS BY QUANDLES

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Abstract. The braid index of a surface-knot F is the minimum number among the degrees of all surface braids whose closures are ambient isotopic to F. We give a lower bound of the braid index of a surface-knot using the colorings by a quandle. As an application, we determine the braid indices of S^2-knots for infinitely many examples and give an infinite series of ribbon surface-knots of genus g whose braid indices are $s+2$ for each pair of integers $g \geq 0$ and $s \geq 1$.

1. Surface braids

Please refer to S. Kamada's book [6] for details of surface braid theory.

Definition 1.1. (surface braid)
Let D_1^2 and D_2^2 be 2-disks and X_m a fixed set of m distinct interior points of D_1^2. Let $pr_i : D_1^2 \times D_2^2 \to D_i^2$ be the projection map to the i-th factor for each i $(i = 1, 2)$. A *surface braid of degree* m (or *surface* m*-braid*) is a compact oriented surface S embedded properly and locally flatly in $D_1^2 \times D_2^2$ such that

(i) the restriction map $pr_2|_S : S \to D_2^2$ is a branched covering map of degree m,
(ii) $\partial S = X_m \times \partial D_2^2 \left(\subset D_1^2 \times \partial D_2^2\right)$, and
(iii) the branched covering $pr_2|_S$ is simple, that is, $|S \cap pr_2^{-1}(y)| = m-1$ or m for each $y \in D_2^2$.

Definition 1.2. (equivalence relation)
Two surface braids S and S' are said to be *equivalent* if there is an ambient isotopy $\{h_t\}_{t\in[0,1]}$ such that

(i) $h_0 = \mathrm{id}, h_1(S) = S'$,
(ii) for each $t \in [0,1]$, h_t is fiber-preserving, that is, there is a homeomorphism $\underline{h}_t : D_2^2 \to D_2^2$ such that $pr_2 \circ h_t = \underline{h}_t \circ pr_2$, and
(iii) for each $t \in [0,1]$, $h_t|_{D_1^2 \times \partial D_2^2} = \mathrm{id}$.

Definition 1.3. (closure)
Let S^2 be a 2-sphere obtained from D_2^2 attaching a 2-disk $\overline{D_2^2}$ along the boundary of D_2^2. A surface braid S of degree m is extended to a closed surface $\widehat{S}$ in $D_1^2 \times S^2$ $\left(= D_1^2 \times (D_2^2 \cup \overline{D_2^2})\right)$ such that

$$\widehat{S} \cap (D_1^2 \times D_2^2) = S \text{ and } \widehat{S} \cap (D_1^2 \times \overline{D_2^2}) = X_m \times \overline{D_2^2} \; .$$

Identifying $D_1^2 \times S^2$ with the tubular neighborhood of a standard 2-sphere in $\mathbb{R}^4$, we assume that $\widehat{S}$ is a closed oriented surface embedded in $\mathbb{R}^4$. We call it the *closure* of S in $\mathbb{R}^4$.

Theorem 1.4. ([4, 10]) *Any oriented surface-link in $\mathbb{R}^4$ is ambient isotopic to the closure of a surface braid of degree m for some m.*

Definition 1.5. The *braid index* of a surface-link F, denoted by $\mathrm{Braid}(F)$, is the minimum number among the degrees of all surface braids whose closures are ambient isotopic to F.

Remark 1.6. (Known results about braid indices)

- The braid index of the trivial n-component S^2-link is n $(n \geq 1)$.
- $\mathrm{Braid}(F) = 1 \Longleftrightarrow F$: the trivial S^2-knot.
- $\mathrm{Braid}(F) = 2$
  $\Longleftrightarrow F$: the trivial 2-component S^2-knot or the trivial Σ_g-knot $(g \geq 1)$.
- $\mathrm{Braid}(F) = 3 \Longrightarrow F$: a ribbon surface-link. (it was shown that the other way does not hold.) ([5])
- There are infinitely many ribbon S^2-knots with braid index 3 ([5]).
- There are infinitely many ribbon S^2-knots with braid index 4 ([7]).

2. Quandles and colorings

Definition 2.1. (quandle)
A *quandle* [1, 2] is a set X equipped with a binary operation $(a, b) \mapsto a * b$ such that (i) $a * a = a$ for any $a \in X$, (ii) the map $*a : X \to X$ $(x \mapsto x * a)$ is bijective for each $a \in X$, and (iii) $(a * b) * c = (a * c) * (b * c)$, for any $a, b, c \in X$. A function $f : X \to Y$ between quandles is a *homomorphism* if $f(a * b) = f(a) * f(b)$ for any $a, b \in X$. For each element $a \in X$, the map $*a : X \to X$ is a quandle automorphism of X by (ii) and (iii), and we denote the inverse map $(*a)^{-1}$ by $\overline{*}\, a$.

Definition 2.2. (knot quandle and coloring)
For $n \geq 0$, let M be an oriented $(n+2)$-dimensional manifold and L an oriented n-dimensional manifold embedded in M properly and locally flatly. Let $N(L)$ denote a tubular neighborhood of L in M. Take a fixed point $z \in E(L) = \mathrm{Cl}(M \backslash N(L))$ and let $Q(M, L, z)$ be the set of homotopy classes of paths $\alpha : [0, 1] \to E(L)$ such that $\alpha(0) \in \partial E(L)$ and $\alpha(1) = z$. A point $p \in \partial E(L)$ lies on a unique meridional circle of $N(L)$. Let m_p be the loop based at p which goes along this meridional circle in a positive direction. The *knot quandle* of L in M, with the base point z, is a quandle consisting of the set $Q(M, L, z)$ with a binary operation defined by

$$[\alpha] * [\beta] = [\alpha \cdot \beta^{-1} \cdot m_{\beta(0)} \cdot \beta] \ .$$

When $M = \mathbb{R}^{n+2}$, we denote $Q(\mathbb{R}^{n+2}, L, z)$ by $Q(L)$ briefly.

Let F be a surface-link and X a finite quandle. A *coloring* of F by X is a quandle homomorphism $c : Q(F) \to X$ from the knot quandle $Q(F)$ to X. We denote by $\mathrm{Col}_X(F)$ the set of all colorings of F by X. Note that the number of the colorings, $|\mathrm{Col}_X(F)|$, is an invariant of the surface-link F.

3. Main results

Theorem 3.1. *Let F be a surface-link which is not a trivial S^2-link. Let X be a finite quandle of order N, where N is a positive integer. If the inequality $|\mathrm{Col}_X(F)| > N^l$ holds for some positive integer l, then we have* $\mathrm{Braid}(F) \geq l + 2$.

By using this theorem and Theorem 4.1, we determine the braid indices of S^2-knots for infinitely many examples.

Theorem 3.2. *For an odd integer $n \geq 3$, let K_n be the S^2-knot obtained from an $(n,2)$-torus knot by Artin's spinning construction. Then we have the following.*

(*i*) *The braid index of an S^2-knot $K_n(s)$ is $s+2$, where $K_n(s)$ is the connected sum of s copies of K_n.*

(*ii*) *The braid index of a Σ_g-knot $K_n(s,g)$ is also $s+2$, where $K_n(s,g)$ is the connected sum of $K_n(s)$ and g copies of a trivial T^2-knot.*

Theorem 3.3. *For each pair of integers $g \geq 0$ and $s \geq 1$, there exists an infinite series of ribbon surface-knots of genus g whose braid indices are $s+2$.*

4. Lemma, Proposition and Theorem

Theorem 4.1. ([7]) *If neither F_1 nor F_2 is a trivial S^2-knot, then the following inequality holds.*

$$\mathrm{Braid}(F_1 \# F_2) \leq \mathrm{Braid}(F_1) + \mathrm{Braid}(F_2) - 2$$

Proposition 4.2. *Let F be a surface-link which is not a trivial S^2-link. Let α be the minimum number of generators of the knot quandle $Q(F)$. Then we have* $\mathrm{Braid}(F) - 1 \geq \alpha$.

Let R_m be a quandle consisting of the set $\{0, 1, \cdots, m-1\}$ with the binary operation defined by $i * j \equiv 2j - i \pmod{m}$, where m is a positive integer. The quandle R_m is called the *dihedral quandle of order m*.

Lemma 4.3. $\left|\mathrm{Col}_{R_m}(K_n(s))\right| \leq m^{s+1}$.
The equality sign holds if and only if n is divided by m.

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