1.1. Definition of the A-polynomial

Cooper-Culler-Gillet-Long-Shalen

"Plane curves associated to character varieties of 3-manifolds."

Invent. Math ('94)

Definition Let
M : *compact* 3-*mfd s.t.* $\partial M = T^2$,
$\pi_1(\partial M) \ni \lambda,\ \mu$
$R := Hom(\pi_1(M),\ SL(2, \mathbf{C}))$
$\cup$
$R_U := \left\{ \rho \in R \,|\, \rho(\lambda) = \begin{pmatrix} l & * \\ 0 & l^{-1} \end{pmatrix}, \rho(\mu) = \begin{pmatrix} m & * \\ 0 & m^{-1} \end{pmatrix} \right\}$

$$Z[l, m] \ni {}^{\exists} f(l, m) = 0,$$

then the A-polynomial of M is defined as

$$A(l, m) = \frac{f(l, m)}{(l-1)}.$$

K : *a knot*, $M = S^3 - N(K)$: *a knot complement*

$$\Rightarrow A_K(l, m)\ :\ A\text{-}polynomial\ of\ K$$

1.2. Computation of the A-polynomial

For example

$K = 3_1$

Figure 1

$$\begin{cases} \pi_1(K) = < \mu, x \mid \mu x \mu = x \mu x > \\ \mu^4 \lambda = \mu x \cdot x \mu \end{cases}$$

After conjugation and after replacing x by x^{-1} if nenessary we can assume that

$$\rho(x) = \begin{pmatrix} m & 0 \\ t & m^{-1} \end{pmatrix}.$$

Then

$$\begin{cases} \rho(\mu)\rho(x)\rho(\mu) = \rho(x)\rho(\mu)\rho(x) \\ \rho(\mu)^4 \rho(\lambda) = \rho(\mu)\rho(x)\rho(x)\rho(\mu) \end{cases}$$

$$\Leftrightarrow \begin{cases} 1 - m^2 + m^4 + m^2 t = 0 \\ -m^4 + l m^4 - t - m^2 t = 0, \end{cases}$$

from these equation we obtain the A-polynomial of K.

$$A_K(l, m) = 1 + l m^6$$

$$A_{4_1} = -m^4 + l(m^8 - m^6 - 2m^4 - m^2 + 1) - l^2 m^4$$

$$
\begin{aligned}
A_{5_2} &= -l^3 + (1 - 2m^2 - 2m^4 + m^8 - m^{10})l^2 \\
&\quad + (-m^4 + m^6 - 2m^{10} - 2m^{12} + m^{14})l - m^{14} \\
A_{6_1} &= -m^8 + (1 - m^2 - 3m^8 - 3m^{10} + 2m^{12})l \\
&\quad + (-1 + 3m^2 + m^4 - 3m^6 - 6m^8 \\
&\qquad\qquad -3m^{10} + m^{12} + 3m^{14} - m^{16})l^2 \\
&\quad + (2 - 3m^2 - 3m^4 - m^{10} + m^{12})m^4 l^3 - m^8 l^4 \\
A_{7_2} &= m^{22} + (1 - m^2 + 3m^{10} + 4m^{12} - 2m^{14})m^8 l \\
&\quad + (-2 + 5m^2 + m^4 - 4m^6 + 6m^{10} \\
&\qquad\qquad +5m^{12} + 2m^{14} - 4m^{16} + m^{18})m^4 l^2 \\
&\quad + (1 - 4m^2 + 2m^4 + 5m^6 + 6m^8 \\
&\qquad\qquad -4m^{12} + m^{14} + 5m^{16} - 2m^{18})l^3 \\
&\quad + (-2 + 4m^2 + 3m^4 - m^{12} + m^{14})l^4 + l^5
\end{aligned}
$$

$$
\begin{aligned}
A_{8_2} = {} & m^{72} + (-5 + 9m^2 + 7m^4 - 3m^6 - m^{12} + 2m^{14} - m^{16})lm^{60} \\
& + (10 - 32m^2 - m^4 + 56m^6 + 17m^8 - 28m^{10} - m^{12} + 14m^{14} - 4m^{16} \\
& \quad - 8m^{18} + 7m^{20} - 2m^{22})l^2 m^{48} \\
& + (-10 + 42m^2 - 24m^4 - 87m^6 + 29m^8 + 143m^{10} + 33m^{12} - 77m^{14} \\
& \quad - 17m^{16} + 29m^{18} - 2m^{20} + m^{22} - 8m^{24} + 5m^{26} - m^{28})l^3 m^{36} \\
& + (5 - 24m^2 + 26m^4 + 36m^6 - 43m^8 - 108m^{10} + 47m^{12} + 192m^{14} \\
& \quad + 47m^{16} - 108m^{18} - 43m^{20} + 36m^{22} + 26m^{24} - 24m^{26} + 5m^{28})l^4 m^{24} \\
& + (-1 + 5m^2 - 8m^4 + m^6 - 2m^8 + 29m^{10} - 17m^{12} - 77m^{14} + 33m^{16} \\
& \quad + 143m^{18} + 29m^{20} - 87m^{22} - 24m^{24} + 42m^{26} - 10m^{28})l^5 m^{12} \\
& + (-2 + 7m^2 - 8m^4 - 4m^6 + 14m^8 - m^{10} - 28m^{12} + 17m^{14} + 56m^{16} \\
& \quad - m^{18} - 32m^{20} + 10m^{22})l^6 m^6 \\
& + (-1 + 2m^2 - m^4 - 3m^{10} + 7m^{12} + 9m^{14} - 5m^{16})l^7 + m^4 l^8
\end{aligned}
$$

2. An ideal triangulation of a knot complement

Y. Yokota

"From the Jones polynomial to the *A*-polynomial of hyperbolic knots."

Int. Sci. (2003)

K : 5_2, M : $S^3 - N(K)$

Figure 2

Now we construct an ideal triangulation of M, where the ideal tetrahedra is given below,

Figure 3

and 3 moduli of a triangle are related as follows.

Figure 4

Then we have $\partial N(K)$.

Figure 5

Then we have following hyperbolicity equations, where the variables l and m represent longitude and meridian.

$$
\begin{aligned}
m^2 &= \frac{z_1}{z_5} = \frac{(1-1/z_3)(1-z_2)}{1-z_4} = \frac{1-1/z_3}{(1-1/z_1)(1-z_5)} \\
l^2 &= (1-z_1)\cdot(1-z_4)\cdot\frac{1}{1-z_4}\cdot\frac{1}{z_3}\cdot\frac{1-z_5}{z_5}\cdot(1-z_1) \\
&\quad \times\frac{1}{1-z_3}\cdot(1-z_1)\cdot\frac{1}{z_5}\cdot\frac{1}{1-z_1}\cdot\frac{1-z_3}{z_3}\cdot\frac{1}{1-z_5} \\
&= \left\{\frac{1}{z_3 z_5}\cdot(1-z_1)\right\}^2
\end{aligned}
$$

If this ideal triangulation determines a hyperbolic structure of M, the product of the moduli around each edge should be 1, that is,

$$1 = z_2 z_4 = z_3 z_4 z_5,$$

which suggest to put

$$z_1 = ym, z_2 = \frac{x}{m}, z_3 = \frac{x}{y}, z_4 = \frac{m}{x}, z_5 = \frac{y}{m},$$

Then we can rewrite the hyperbolicity equations

$$m^2 = \frac{(1 - y/x)(1 - x/m)}{1 - m/x} = \frac{1 - y/x}{(1 - y/m)(1 - 1/ym)}$$

and

$$l = \frac{m}{x} \cdot (1 - ym).$$

¿From these equations we have

$$\begin{aligned} A_K(l, m) = & -l^3 + (1 - 2m^2 - 2m^4 + m^8 - m^{10})l^2 \\ & +(-m^4 + m^6 - 2m^{10} - 2m^{12} + m^{14})l - m^{14}. \end{aligned}$$

3. A formula for some infinite knot families

J. Hoste, P. D. Shanahan

"A formula for the A-polynomial of twist knots"

(2002)

The twist knot is the knot shown in the following picture.

Figure 6

Theorem

The A-polynomial $A_n(l,m)$ of the twist knot is given recursively by

$$A_n(l,m) = xA_{n-n/|n|}(l,m) - yA_{n-2n/|n|}(l,m),$$

where x, y and initial conditions A_{-1}, A_0, A_1, A_2 are the polynomial of l and m.

N. Tamura, Y. Yokota

"A formula for the A-polynomials of $(-2, 3, 2n+1)$-pretzel knots."

Tokyo Math. to appear.

The pretzel knot is the knot shown in the following.

Figure 7

This is $(-2, 3, 7)$-pretzel knot. Then let K_n be the knot shown in the following picture.

Figure 8

If $n = 0, 1, 2, 3$, the knot K_n are the following knots.

$K_0 = 5_1$

$$A_{K_0}(l, m) = 1 + lm^{10},$$

$K_1 = 8_{19}$

$$A_{K_1}(lm^{-4}, m) = 1 + lm^8,$$

$K_2 = 10_{124}$

$$A_{K_2}(lm^{-8}, m) = (1 + lm^7)(1 - lm^7),$$

$K_3 = (-2, 3, 7)$-pretzel knot

$$\begin{aligned} &A_{K_3}(lm^{-12}, m) \\ &\quad = 1 - (1 - 2m^2 + m^4)\, m^4\, l - (2 + m^2)\, m^{12}\, l^2 \\ &\qquad + (1 + 2m^2)\, m^{24}\, l^4 + (1 - 2m^2 + m^4)\, m^{30}\, l^5 \\ &\qquad - m^{38}\, l^6. \end{aligned}$$

Main Theorem

We can define $A_n(lm^{-4n}, m)$ recursively by

$$\frac{A_n}{B_n} = \frac{\gamma}{\alpha}\,\frac{A_{n-1}}{B_{n-1}} + (2 + \frac{2\gamma}{\alpha} + \frac{\beta^2}{\alpha})\,\frac{A_{n-2}}{B_{n-2}} + \frac{\gamma}{\alpha}\,\frac{A_{n-3}}{B_{n-3}} - \frac{A_{n-4}}{B_{n-4}},$$

where B_n is

$$\begin{cases} -l^2(lm^8)^{3+n}(1-m^2)^n(1+lm^6)^{3+n} & (n > 3), \\ -(lm^8)^{-(2+n)}(1-m^2)^{-(1+n)}(1+lm^6)^{2-n} & (n < 0) \end{cases}$$

and

$$\begin{aligned} \alpha &= lm^8(1-m^2)(1+lm^6), \\ \beta &= m^2 - (1-2m^2)l\,m^6 - (2+m^2)\,l^2\,m^{16} - l^3\,m^{22}, \\ \gamma &= -(1+m^4) - (2+m^2-m^4)\,l\,m^8 \\ &\qquad + (-l+m^2+2m^4)\,l^2\,m^{12} + (1+m^4)\,l^3\,m^{20}. \end{aligned}$$

♠ The outline of the proof.

Figure 9

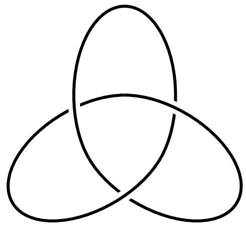


Figure 1:

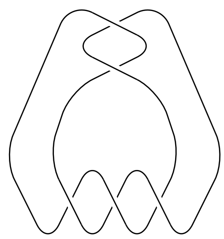


Figure 2:

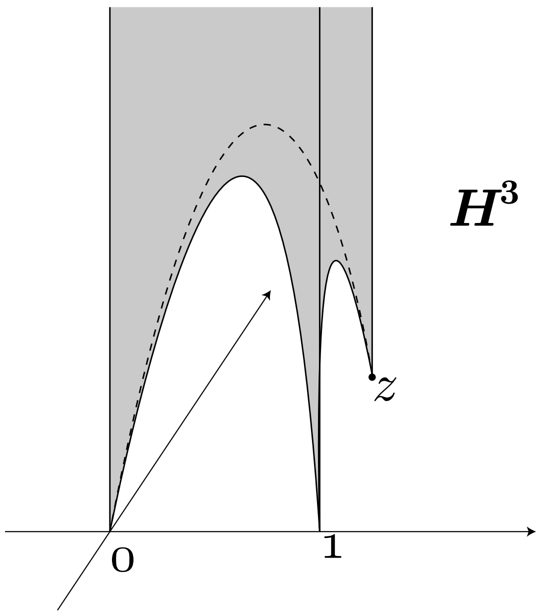


Figure 3:

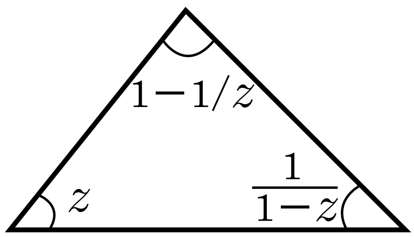
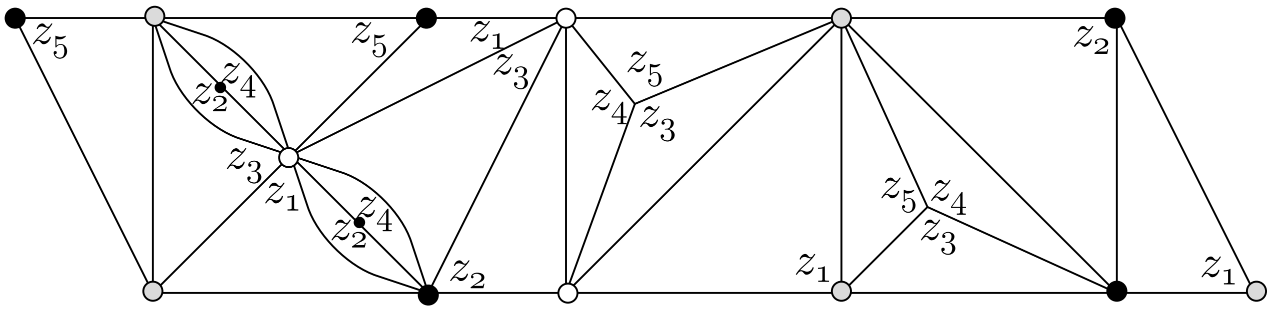


Figure 4:



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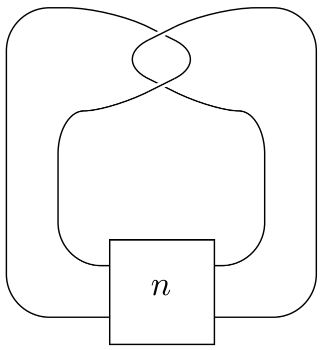

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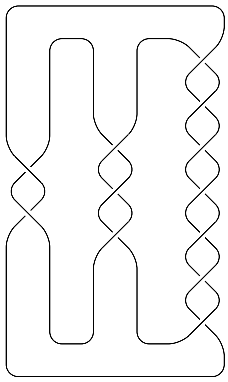

Figure 7:

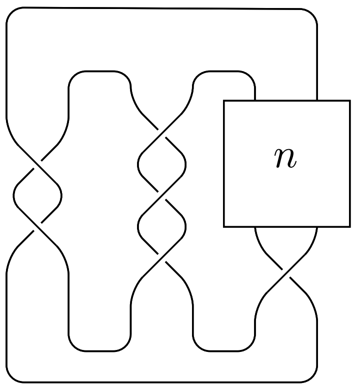

Figure 8:

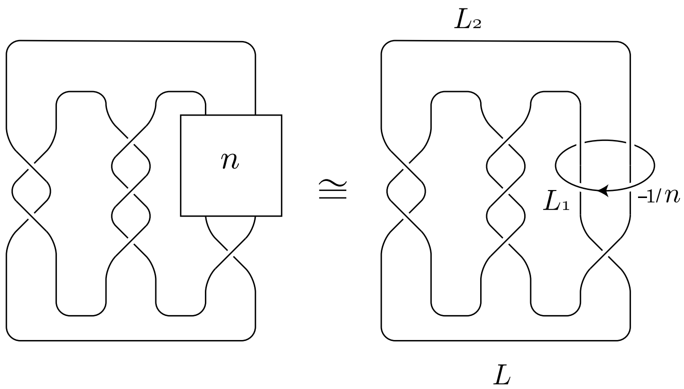

Figure 9: