DELTA LINK-HOMOTOPY ON SPATIAL GRAPHS

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Abstract. Delta link-homotopy is an equivalence relation on oriented links generated by delta moves on the same component and ambient isotopies, and extended to spatial graphs naturally. In this talk, we will explain the content of the papers [18, 19, 20]. We also refer the reader to [22, 23, 24] for their outlines in Japanese.
In this note we will discuss about

- the relation between delta link-homotopy and the other equivalence relations and
- (complete) classifications of spatial embeddings of certain graphs up to delta link-homotopy.

Acknowledgment.

This lecture note is based on the notes made by Kazuhiro Ichihara. All of the figures in this note were illustrated by Reiko Shinjo. The author wishes to express his deep gratitude to them.

0. Equivalence relations on spatial graphs

Throughout this talk, we only consider a finite graph without free vertices (i.e., vertices with valency 0, 1). We always regard a graph G as a 1-dimensional CW- or simplicial complex as usual. We call an embedding $f : G \to S^3$ a *spatial embedding of* G, or simply, a *spatial graph*.

Example 0.1. Throughout the following, all vertices and edges of graphs will be assumed to be labeled with numbers.

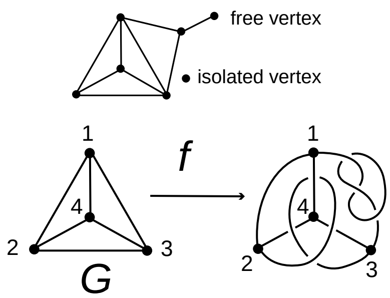

Fig. 0.1

In this section we recall basic definitions, which will be used throughout the talk. First we give a summary of known equivalence relations for spatial graphs.

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Definition 0.2 (equivalence relations, [32])**.** Let f, g be spatial embeddings of a graph G.

(1) f and g are *ambient isotopic* if there exists an orientation preserving homeomorphism $\Phi : S^3 \to S^3$ such that $f \circ \Phi = g$ holds.

It is known by [6, 35] that this is equivalent to that f and g are transformed into each other by the *Reidemeister moves*; that are (I), (II), (III) (original) Reidemeister moves for knots, and (IV), (V), see Fig. 0.2.

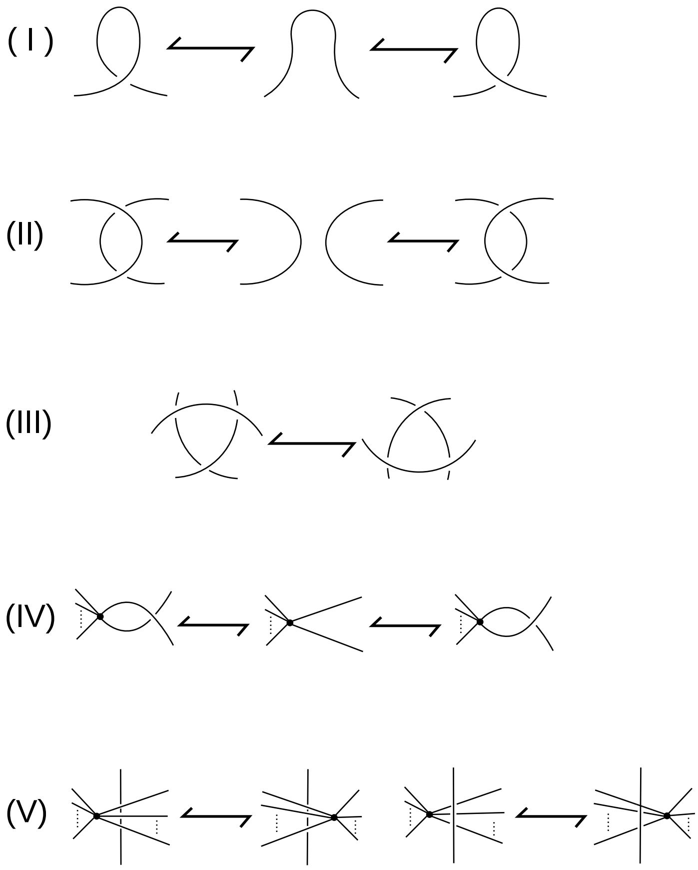

Fig. 0.2. Reidemeister moves

This is also equivalent to that there exists a level preserving locally flat embedding $\Phi : G \times I \to S^3 \times I$ between f and g. Here $\Phi : G \times I \to S^3 \times I$ is said to be
(a) *between f and g* if there is a real number $\varepsilon > 0$ such that $\Phi(x, t) = (f(x), t)$ for any $x \in G$, $0 \leq t \leq \varepsilon$ and $\Phi(x, t) = (g(x), t)$ for any $x \in G$, $1 - \varepsilon \leq t \leq 1$,
(b) *locally flat* if every point $p \in \Phi(G \times I)$ has a neighborhood N such that $(N, N \cap \Phi(G \times I))$ is pairwise homeomorphic to the standard disk pair (D^4, D^2) or $(D^3, X_n) \times I$, where (D^3, X_n) denotes the pair as illustrated in Fig. 0.3 and

(c) *level preserving* if there is a map $\phi_t : G \to S^3$ for each $t \in I$ such that $\Phi(x,t) = (\phi_t(x), t)$ for any $x \in G$, $t \in I$.

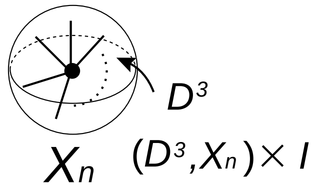

Fig. 0.3

(2) f and g are *cobordant*, denoted by $f \overset{(\mathrm{Cob})}{\sim} g$, if there exists a locally flat embedding $\Phi : G \times I \to S^3 \times I$ between f and g (see Fig. 0.4).

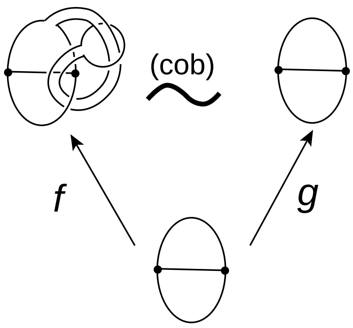

Fig. 0.4. Cobordant spatial embeddings

(3) f and g are *isotopic*, denoted by $f \overset{(\mathrm{Iso})}{\sim} g$, if there exists a level preserving embedding $\Phi : G \times I \to S^3 \times I$ between f and g. Isotopy on spatial graphs is an application of Alexander trick, see Fig. 0.5.

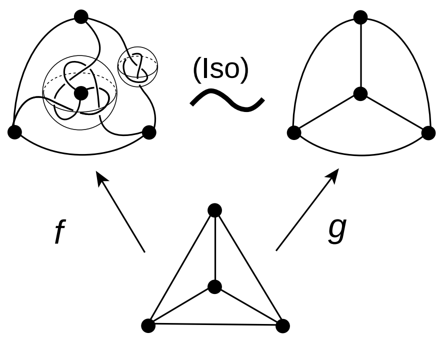

Fig. 0.5. Isotopic spatial embeddings

(4) f and g are *I-equivalent* if they are "cobordant" or "isotopic". Precisely they are I-equivalent if there exists an embedding $\Phi : G \times I \to S^3 \times I$ between f and g.

(5) f and g are *edge-homotopic*, denoted by $f \overset{\text{(EH)}}{\sim} g$, if f and g are transformed into each other by self crossing changes and ambient isotopic, see Fig. 0.6.

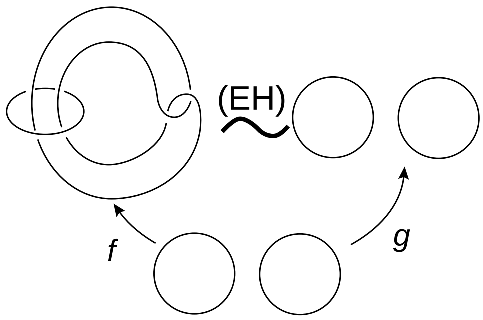

Fig. 0.6. Edge-homotopic spatial embeddings (2-component links)

Here *self crossing changes* mean the crossing changes on the same spatial edge as illustrated in Fig. 0.7.

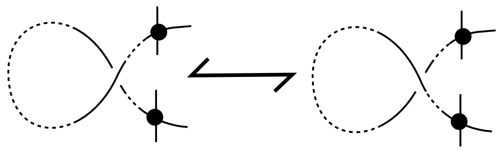

Fig. 0.7. Self crossing change

Remark. Edge-homotopy on spatial graphs is a generalization of link-homotopy on links in the sense of Milnor [10].

A graph G is said to be *planar* if there exists an embedding $f : G \to S^2$, and is said to be *non-planer* otherwise. By the well-known Kuratowski's theorem, G is non-planar if and only if it contains a subgraph which is homeomorphic to K_5 or $K_{3,3}$ as illustrated in Fig. 0.8

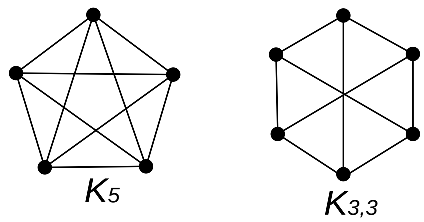

Fig. 0.8

Definition 0.3. Suppose that G is planar. Let f be a spatial embedding of G.

(1) f is *trivial* if f is ambient isotopic to an embedding $h : G \to S^2 \subset S^3$.

(2) f is *slice* if f is cobordant to the trivial embedding.

Remark. The well-definedness of the trivial embedding was shown in [8]: Any two embeddings of G into $S^2 \subset S^3$ are actually ambient isotopic.

1. Delta edge-homotopy, Delta vertex-homotopy

In this section, we introduce two more equivalence relations, which can be regarded as a natural extension of delta link-homotopy on links. These are the main subjects in this talk.

A *delta move* is a local move on links as illustrated in Fig. 1.1. This move was introduced in [9, 12] and showed that it is an unknotting operation for knots. This is naturally extended to a local move to spatial graphs.

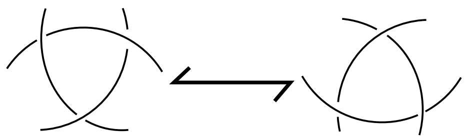

Fig. 1.1. Delta move

Remark. The delta move is not an unlinking operation for links. Because it keeps linking numbers of links invariant.

Adding certain restrictions, let us introduce two similar local moves. A *self delta move* is defined to be the delta move on the same spatial edge as illustrated in Fig. 1.2.

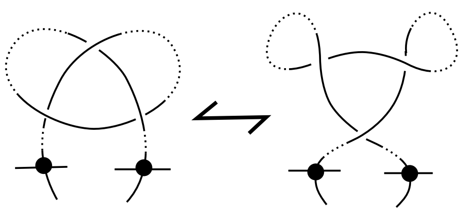

Fig. 1.2. Self delta move

A *quasi adjacent-delta move* is defined to be the delta move on exactly two adjacent spatial edges as illustrated in Fig. 1.3.

By using these moves, two equivalence relations are defined.

Definition 1.1. ([18]) Two spatial embedding $f, g : G \to S^3$ are

(1) *delta edge-homotopic*, denoted by $f \overset{(\mathrm{DEH})}{\sim} g$, if f and g are transformed into each other by self delta moves and ambient isotopies,

(2) *delta vertex-homotopic*, denoted by $f \overset{(\mathrm{DVH})}{\sim} g$, if f and g are transformed into each other by quasi adjacent-delta moves and ambient isotopies,

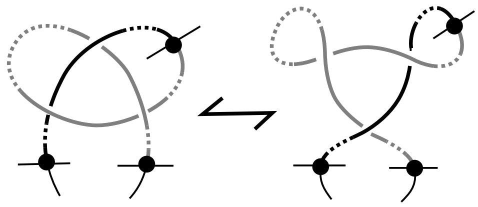

Fig. 1.3. Quasi adjacent-delta move

Remark.

- In the case of $G = S^1 \amalg \cdots \amalg S^1$, delta edge-homotopy and delta vertex-homotopy are equivalent as equivalence relations, for they are natural extension of *delta link-homotopy* on links [29, 30, 17, 13, 14, 15].
- In general, a self delta move is not an unlinking operation. However it actually is for ribbon links and for 2-component boundary links [29, 30]. It is still open for boundary links with three or more components.

A motivation to study these moves, at least for the author, is to classify spatial graphs without considering 'local knots'. In some sense, the properties of spatial graphs invariant under self crossing changes or self delta moves must be essential in spatial graph theory independent from knot theory.

2. Relation to the other equivalence relations

Theorem 2.1. *The following implications hold* [32].

$$\begin{tikzcd}
 & \boxed{\textit{Cobordism}} \arrow[dr] & & \\
\boxed{\textit{Ambient isotopy}} \arrow[ur] \arrow[dr] & & \boxed{\textit{I-equivalence}} \arrow[r] & \boxed{\textit{Edge-homotopy}} \\
 & \boxed{\textit{Isotopy}} \arrow[ur] & &
\end{tikzcd}$$

Moreover the following implications hold [18].

$$\begin{tikzcd}
\boxed{\textit{Ambient isotopy}} \arrow[rr] \arrow[dr] & & \boxed{\textit{Isotopy}} \arrow[rr] \arrow[dr] & & \boxed{\textit{Edge-homotopy}} \\
 & \boxed{\textit{Delta edge-homotopy}} \arrow[rr] & & \boxed{\textit{Delta vertex-homotopy}} \arrow[ur] &
\end{tikzcd}$$

In the theorems above, $\boxed{(1)} \to \boxed{(2)}$ means that $f \overset{(1)}{\sim} g$ implies $f \overset{(2)}{\sim} g$. It is remarked that there are no implications between ones where no vectors are drawn, and no converses of the implications described above do not hold.

Outline of proof. We can see that $\boxed{\text{(DEH)}} \to \boxed{\text{(DVH)}} \to \boxed{\text{(EH)}}$ by Fig. 2.1.

To show that $\boxed{\text{(Iso)}} \to \boxed{\text{(DVH)}}$, we use the next claim.

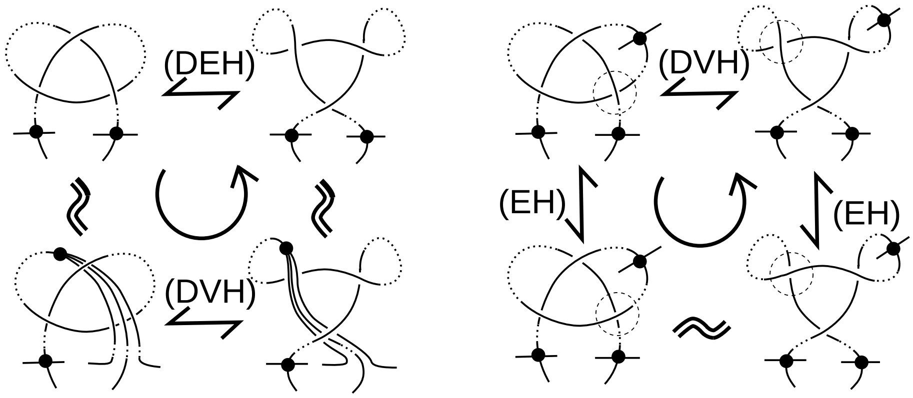

Fig. 2.1

Claim. *Each of the moves as illustrated in Fig. 2.2 is realized by delta moves and ambient isotopies.*

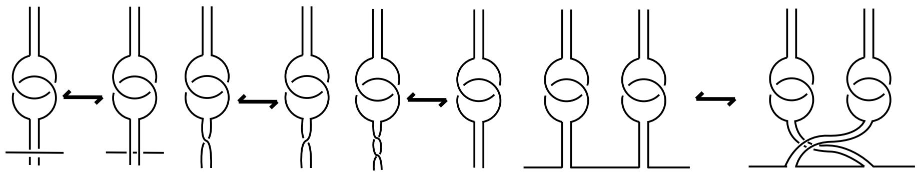

Fig. 2.2

Proof. See Fig. 2.3 for example.

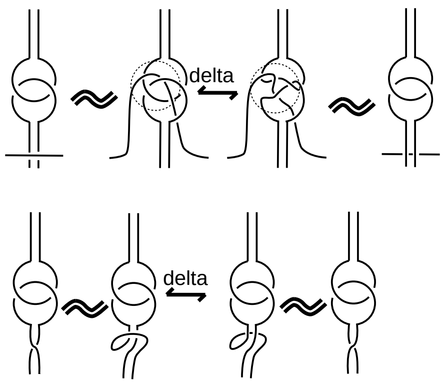

Fig. 2.3

□

We note that crossing changes can be regarded as a 'band sum of Hopf links', see Fig. 2.4. With the claim above and the fact that isotopy implies edge-homotopy, Fig. 2.5 illustrates the implication $\boxed{\text{(Iso)}} \rightarrow \boxed{\text{(DVH)}}$.

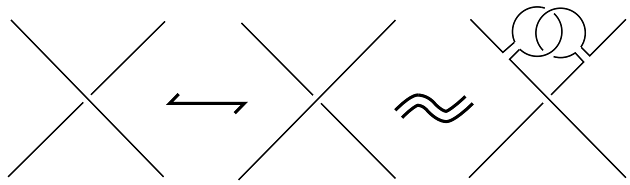

Fig. 2.4

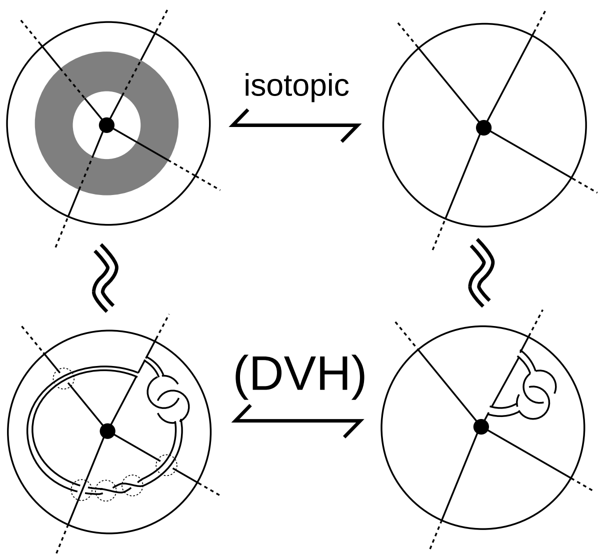

Fig. 2.5

Please see [18] for detailed proofs and proofs of other parts. □

Together with ∽'s result [32], we have the following corollary.

Corollary 2.2. *Let G be a finite graph. Then the following are equivalent.*

(1) *Every pair of spatial embeddings of G are isotopic.*
(2) *Every pair of spatial embeddings of G are I-equivalent.*
(3) *Every pair of spatial embeddings of G are delta vertex-homotpic.*
(4) *Every pair of spatial embeddings of G are edge-homotpic.*
(5) *G is a generalized bouquet, i.e., G does not contain a subgraph which is homeomorphic to disjoint cycles, K_4 or D_3 as illustrated in Fig. 2.6.*

Recently the author obtained the following related result.

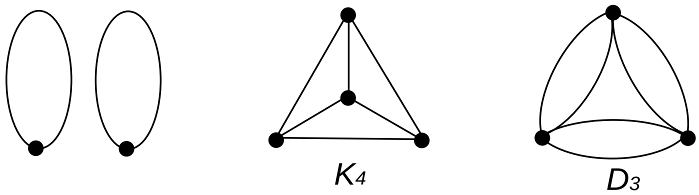

Fig. 2.6

Theorem 2.3. ([21]) *Let G be a finite graph. Then the following are equivalent.*

(1) *Every pair of spatial embeddings of G are delta edge-homotpic.*
(2) *G does not contain a subgraph which is homeomorphic to a θ-curve or disjoint cycles.*
(3) *G is a bouquet as illustrated in Fig. 2.7.*

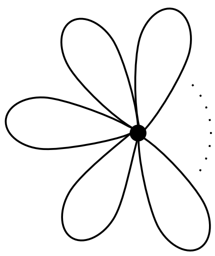

Fig. 2.7

3. Delta edge- & Delta vertex-homotopy invariants

We start with an example of a delta edge-homotopy invariant. A subgraph of G is called a *cycle* if it is homeomorphic to S^1. A cycle of a graph is called a *k-cycle* if it contains exactly k edges.

Example. Set G denotes a θ-curve, and give labels to its vertices and edges as in Fig. 3.1. We denote the three cycles $e_2 \cup e_3$, $e_3 \cup e_1$ and $e_1 \cup e_2$ of G by γ_1, γ_2 and γ_3 respectively.

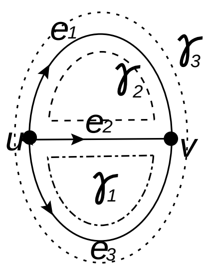

Fig. 3.1

Given spatial embedding $f : G \to S^3$, we define

$$\widetilde{\alpha}(f) \equiv \sum_{i=1}^{3} a_2(f(\gamma_i)) \pmod 2$$

where a_2 denotes the second coefficient of the *Conway polynomial* of the knot. This $\widetilde{\alpha}$ become a delta edge-homotopy invariant, showed as follows. Recall that $a_2(K_+) - a_2(K_-) = 1$ holds for the knots K_+ and K_- as illustrated in Fig. 3.2 [26].

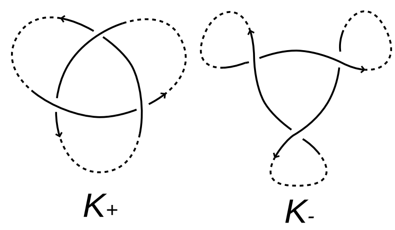

Fig. 3.2

Assume that a θ-curve g is obtained from f by a single self delta move on $f(e_1)$ as illustrated in Fig. 3.3. Then we have that

$$\widetilde{\alpha}(f) - \widetilde{\alpha}(g) \equiv \sum_{i=2}^{3} \{a_2(f(\gamma_i)) - a_2(g(\gamma_i))\} = 2 \equiv 0 \pmod 2.$$

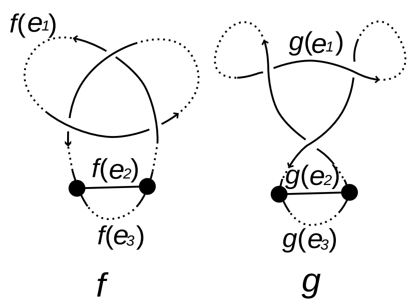

Fig. 3.3

This implies that $\widetilde{\alpha}$ is a delta edge-homotopy invariant.

Now let h be a trivial θ-curve and f a θ-curve as illustrated in Fig. 3.4. Then, by direct calculations, we have that $\widetilde{\alpha}(h) \equiv 0$ and $\widetilde{\alpha}(f) \equiv 1 \pmod 2$. Thus we can conclude that $h \overset{(\mathrm{DEH})}{\not\sim} f$. However it depends only upon the *Arf invariant* of constituent knots, so, it seems to be not strong enough. For example, we cannot distinguish any *almost trivial theta curve* (i.e., spatial embedding with no non-trivial knots) from the trivial one by $\widetilde{\alpha}$.

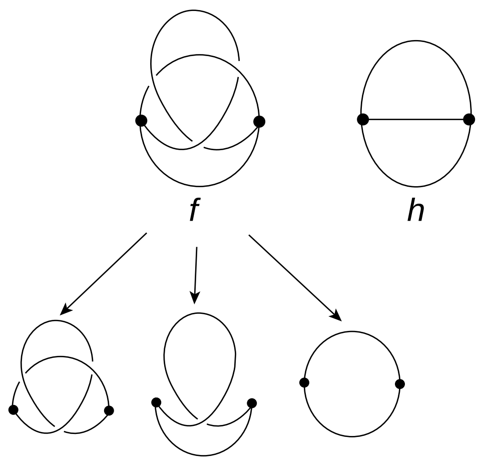

Fig. 3.4

We generalize this invariant in the following way. Let $\Gamma(G)$ be the set of all cycles of a graph G and $E(G)$ the set of all edges of G. For an edge $e \in E(G)$, we set

$$\Gamma_e(G) := \{\gamma \in \Gamma(G) \mid \gamma \supset e\}.$$

Also, for edges $e_1, e_2 \in E(G)$, we set

$$\Gamma_{e_1,e_2}(G) := \{\gamma \in \Gamma(G) \mid \gamma \supset e_1, e_2\}.$$

Definition 3.1. Let $\omega : \Gamma(G) \to \mathbf{Z}_m$ be a map, which we call a *weight*, where $\mathbf{Z}_m$ denotes the subset $\{0, 1, \ldots, m-1\}$ of the infinite cyclic group $\mathbb{Z}$ (we admit $m = 0$, and then $\mathbf{Z}_0$ denotes $\mathbb{Z}$).

(1) ω is *weakly balanced on an edge* $e \in E(G)$ if

$$\sum_{\gamma \in \Gamma_e(G)} \omega(\gamma) \equiv 0 \pmod{m}.$$

is satisfied.

(2) ω is *weakly balanced on a pair of adjacent edges* $e_1, e_2 \in E(G)$ if

$$\sum_{\gamma \in \Gamma_{e_1,e_2}(G)} \omega(\gamma) \equiv 0 \pmod{m}$$

is satisfied.

Given spatial embedding $f : G \to S^3$ and a weight $\Gamma(G) \to \mathbf{Z}_m$, we define

$$\widetilde{\alpha}_\omega(f) \equiv \sum_{\gamma \in \Gamma(G)} \omega(\gamma) a_2(f(\gamma)) \pmod{m}.$$

Theorem 3.2. *Let $f : G \to S^3$ be a spatial embedding of a finite graph G and $\Gamma(G) \to \mathbf{Z}_m$ a weight.*

(1) *If ω is weakly balanced on every edge in $E(G)$, then $\widetilde{\alpha}_\omega$ is a delta edge-homotopy invariant.*

(2) *If ω is weakly balanced on every pair of adjacent edges in $E(G)$, then $\widetilde{\alpha}_\omega$ is a delta vertex-homotopy invariant.*

The proof is similar to the discussion given in the first example in this section, and so we omit it. The invariant $\widetilde{\alpha}_\omega$ is an extension of $\widetilde{\alpha}$: In fact $\widetilde{\alpha}$ is obtained by choosing a weight $\omega : \Gamma(G) \to \mathbf{Z}_2$ with $\omega(\gamma) = 1$ for any $\gamma \in \Gamma(G)$ if G is a θ-curve.

Next we will construct another invariant by using an order three Vassiliev invariant of knots. In the following the graph G is assumed to be oriented, i.e., each edge of G are assumed to be oriented. We set the orientation of each cycle $\gamma \in \Gamma_e(G)$ induced from that of e. Also we set the orientation of each cycle $\gamma \in \Gamma_{e_1,e_2}(G)$ induced from that of e_1.

Definition 3.3 ([33])**.** Let $\omega : \Gamma(G) \to \mathbf{Z}_m$ be a weight.

(1) ω is *balanced on an edge* $e \in E(G)$ if
$$\sum_{\gamma \in \Gamma_e(G)} \omega(\gamma) \cdot \gamma = 0 \text{ in } H_1(G; \mathbb{Z}/m\mathbb{Z})$$
is satisfied.

(2) ω is *balanced on a pair of adjacent edges* $e_1, e_2 \in E(G)$ if
$$\sum_{\gamma \in \Gamma_{e_1,e_2}(G)} \omega(\gamma) \cdot \gamma = 0 \text{ in } H_1(G; \mathbb{Z}/m\mathbb{Z})$$
is satisfied.

Remark. If a weight ω is balanced on every edge in $E(G)$, the invariant $\widetilde{\alpha}_\omega$ is equal to the *α-invariant* α_ω in [33]. Also note that a balanced weight is weakly balanced.

Now, for a spatial embedding $f : G \to S^3$, we define
$$n_\omega(f) \equiv \frac{1}{18} \sum_{\gamma \in \Gamma(G)} \omega(\gamma) \cdot V^{(3)}_{f(\gamma)}(1) \pmod{m},$$
where $V_L(t)$ denotes the *Jones polynomial*[1] of the link L and
$$V^{(3)}_L(1) = \left.\frac{d^3}{dt^3}\right|_{t=1} V_L(t).$$

Remark. As we will say later, the value $\frac{1}{18}V^{(3)}_K(1)$ is always an integer. It is known that this $\frac{1}{18}V^{(3)}_K(1)$ is a basis of Vassiliev invariants of knots of order three.

Theorem 3.4. *Let $f : G \to S^3$ be a spatial embedding of a finite graph G and $\Gamma(G) \to \mathbf{Z}_m$ a weight.*

(1) *If ω is balanced on every edge in $E(G)$, then n_ω is a delta edge-homotopy invariant.*

(2) *If ω is balanced on every pair of adjacent edges in $E(G)$, then n_ω is a delta vertex-homotopy invariant.*

Key of the proof. Let K_+, K_- and K_0 be two knots and a 3-component link as illustrated in Fig. 3.5.

Then we have that
$$\frac{1}{18}V^{(3)}_{K_+}(1) - \frac{1}{18}V^{(3)}_{K_-}(1) = 2\mathrm{Lk}(K_0) - 1,$$

[1] We calculate the Jones polynomial of a link by the skein relation $tV_{J_+}(t) - t^{-1}V_{J_-}(t) = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})V_{J_0}(t)$.

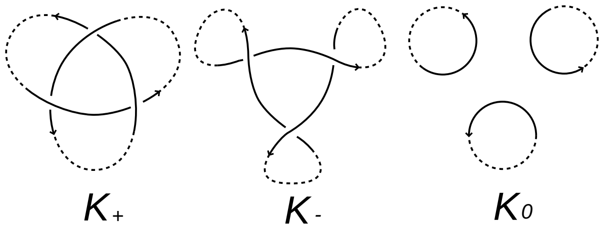

Fig. 3.5

where Lk denotes the *total linking number* (i.e., sum of pairwise linking numbers) of the link K_0. This is a corollary of more general formula obtained in [5]. Since the delta move is an unknotting operation, this formula implies that $\frac{1}{18}V_K^{(3)}(1)$ is an integer for any knot K.

Using this, the theorem follows from direct calculations. □

Remark. The original idea of the theorem above is the construction of an edge-homotopy invariant of spatial graphs in [33] by using the formula $a_2(J_+)-a_2(J_-) = lk(K_0)$, where J_+, J_- and J_0 are two knots and a 2-component link as illustrated in Fig. 3.6.

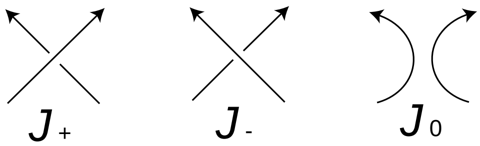

Fig. 3.6

Example 3.5. Let G be K_4 and $m \in \mathbb{Z}$. Let f_m be a spatial embedding of G as illustrated in Fig. 3.7 and h the trivial one.

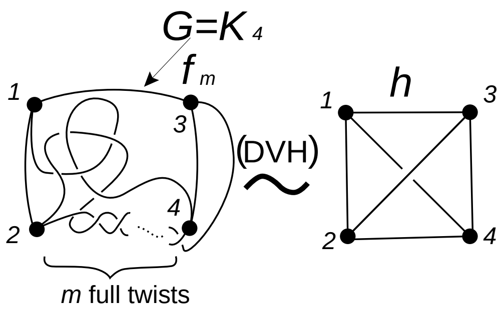

Fig. 3.7

Let $\omega_1 : \Gamma(K_4) \to \mathbf{Z}_4$ be the weight defined by $\omega(\gamma) = 1$ for every cycle $\gamma \in \Gamma(K_4)$. It is easily checked that this ω_1 is weakly balanced on every edge in $E(K_4)$.

The image $f_m(K_4)$ contains two non-trivial knots J_1 and J_2 as illustrated in Fig. 3.8.

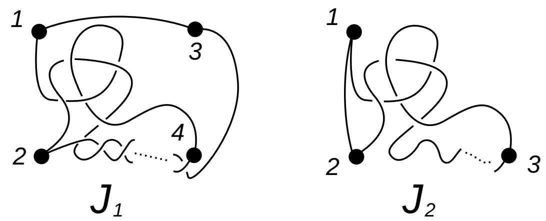

Fig. 3.8

We can calculate that $a_2(J_1) = 1$ and $a_2(J_2) = 1$. Thus we have

$$\widetilde{\alpha}_{\omega_1}(f_m) \equiv 2 \pmod 4$$

and also obviously we have

$$\widetilde{\alpha}_{\omega_1}(h) \equiv 0 \pmod 4.$$

These conclude that $f_m \overset{\text{(DEH)}}{\not\sim} h$ for any $m \in \mathbb{Z}$.

Remark. It is easy to see that $f_m \overset{\text{(DVH)}}{\sim} h$ for any $m \in \mathbb{Z}$. Thus $\widetilde{\alpha}_{\omega_1}$ can detect the difference between delta edge-homotopy and delta vertex-homotopy.

Next we consider the weight $\omega_2 : \Gamma(K_4) \to \mathbb{Z}$ defined by

$$\Gamma(K_4) \ni \gamma \mapsto \begin{cases} 1 & \gamma : \text{ 3-cycle} \\ -1 & \gamma : \text{ 4-cycle.} \end{cases}$$

This ω_2 is also checked to be balanced on every edge in $E(K_4)$. By calculations, we obtain $V^{(3)}_{J_1}(1) = 36m - 18$ and $V^{(3)}_{J_2}(1) = -18$, and so,

$$n_{\omega_2}(f_m) = \frac{1}{18}(-18 - 36m + 18) = -2m.$$

This implies that $f_i \overset{\text{(DEH)}}{\not\sim} f_j$ for any $i \neq j$. So there are infinitely many spatial embeddings of K_4 up to delta edge-homotopy which are mutually delta vertex-homotopic.

Example 3.6. Let $G = K_5$ and $m \in \mathbb{N} \cup \{0\}$. Let f_m be a spatial embedding of G as illustrated in Fig. 3.9. Note that $f_m \overset{\text{(EH)}}{\sim} f_0$, which is achieved by the self crossing changes as in Fig. 3.10.

Let $\omega : \Gamma(K_5) \to \mathbb{Z}$ be the weight defined by

$$\Gamma(K_5) \ni \gamma \mapsto \begin{cases} 0 & \gamma : \text{ 3-cycle} \\ -1 & \gamma : \text{ 4-cycle} \\ 1 & \gamma : \text{ 5-cycle.} \end{cases}$$

Then we can check that this ω is balanced on every pair of adjacent edges in $E(K_5)$. After calculations, one can get $n_\omega(f_m) = -2m$, and therefore, $f_i \overset{\text{(DVH)}}{\not\sim} f_j$ for any $i \neq j$. By Theorem 2.1, this also implies that $f_i \overset{\text{(Isot)}}{\not\sim} f_j$ for any $i \neq j$.

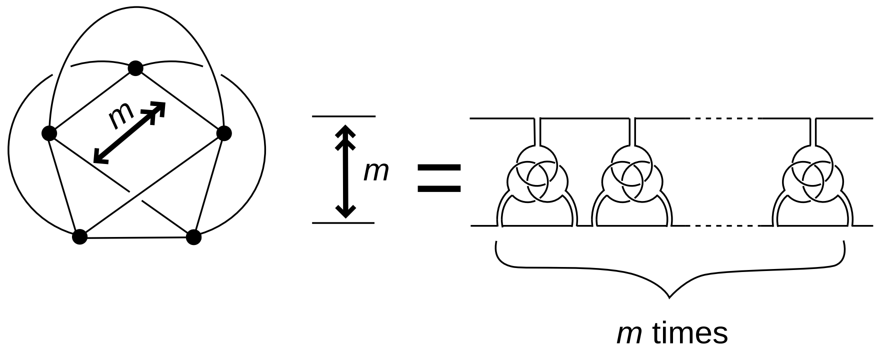

Fig. 3.9

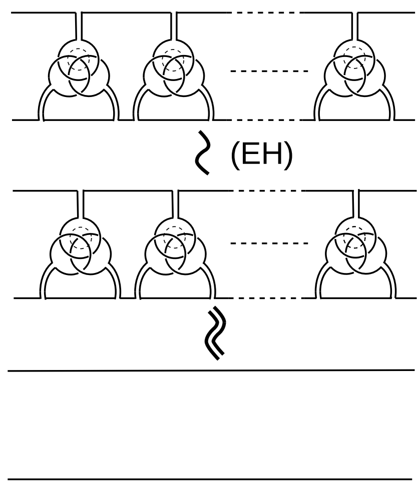

Fig. 3.10

Since all f_m's are mutually edge-homotopic, n_ω can detect the difference between edge-homotopy and delta vertex-homotpy. Moreover there are infinitely many spatial embeddings of K_5 up to delta vertex-homotopy (resp. isotopy) which are mutually edge-homotopic.

4. Delta vertex- & Edge-homotopy classication of spatial embeddings of K_4

By Corollary 2.2, disjoint cycles, K_4, and D_3 are the 'smallest' graphs admitting non-trivial spatial embeddings up to delta vertex-homotopy. This also holds for isotopy, I-equivalence and edge-homotopy. Thus it is natural to ask:

Question. *Can we classify spatial embeddings of such 'smallest' graphs up to delta vertex-homotopy?*

We note that if G is the disjoint cycles then delta vertex-homotopy coinsides with edge-homotopy and all spatial embeddings of G can be classified completely by the linking number [17]. In this section we give an answer to the case of K_4.

Now we put $G = K_4$ or D_3 with labels depicted as in Fig. 4.1.

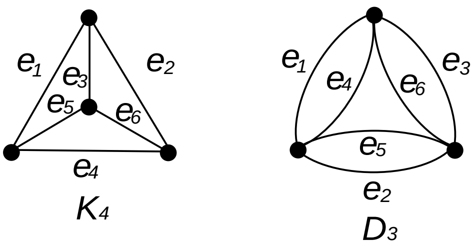

Fig. 4.1

To classify spatial embeddings of K_4 up to delta vertex-homotopy, we use the weight $\omega_2 : \Gamma(K_4) \to \mathbb{Z}$ defined in Example 3.5. To simplify the notation, here we use ω in stead of ω_2. This weight is checked to be balanced on every edge in $E(K_4)$.

Remark. The weight ω is actually shown to be the unique $\mathbb{Z}$-valued balanced weight up to multiplications of constant. Such a 'canonical' balanced weight exists for D_3, but here we omit the details.

Since ω is balanced, in particular, is weakly balanced, on every edge in $E(K_4)$, we can consider the invariant $\widetilde{\alpha}_\omega$. In this case, as noted in the remark just after Definition 3.3, the invariant $\widetilde{\alpha}_\omega$ is equal to $\backsim$'s α-invariant α_ω, which is an edge homotopy invariant.

To state our classification theorem, we need to introduce one more equivalence relation. An *adjacent-delta move* is defined to be the delta move on exactly three adjacent spatial edges as illustrated in Fig. 4.2.

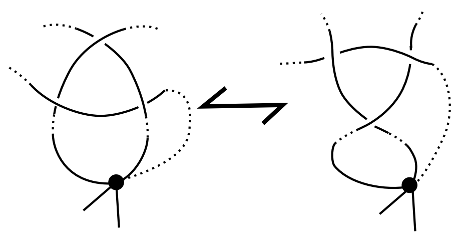

Fig. 4.2. Adjacent-delta move

Remark. An adjacent-delta move does not change the types of the knots included in the spatial graph.

Two spatial embedding $f, g : G \to S^3$ are Δ-*homotopic* if f and g are transformed into each other by quasi adjacent-delta moves, adjacent-delta moves and ambient isotopies. Then we have the following theorem.

Theorem 4.1. *Let $G = K_4$ or D_3, and $\omega : \Gamma(G) \to \mathbb{Z}$ the 'canonical' balanced weight. Then two spatial embedding $f, g : G \to S^3$ are Δ-homotopic if and only if $\alpha_\omega(f) = \alpha_\omega(g)$.*

It can be seen that, for a trivalent graph, an adjacent-delta move is realized by a sequence of quasi adjacent-delta moves (see Fig. 4.3). This indicates that, for a

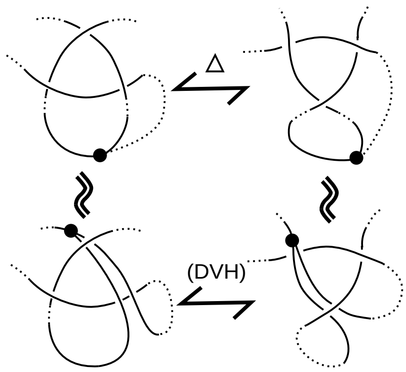

Fig. 4.3

trivalent graph G, in particular, for $G = K_4$, two spatial embedding $f, g : G \to S^3$ are Δ-homotopic only if $f \overset{\text{(DVH)}}{\sim} g$. Consequently we obtain:

Corollary 4.2. *Let $f, g : K_4 \to S^3$ be two spatial embeddings of K_4. Then the following are equivalent.*

(1) f *and* g *are* Δ*-homotopic.*

(2) $f \overset{\text{(DVH)}}{\sim} g$.

(3) $f \overset{\text{(EH)}}{\sim} g$.

(4) $\alpha_\omega(f) = \alpha_\omega(g)$.

Outline of the proof of Theorem 4.1. The 'only if' part is almost clear: It follows from the fact that α_ω depends essentially upon a_2 and that an adjacent-delta move does not change the types of the knots included in the spatial graph.

To prove the 'if' part, we construct a complete system of the representative for the Δ-homotopy types of spatial embeddings of G. For $G = K_4$, we prepare spatial embeddings h_m ($m \in \mathbb{Z}$) as illustrated in Fig. 4.4.

Then we can show that if $\alpha_\omega(f) = m$, then f is Δ-homotopic to h_m. We refer the reader to [19] for the details. This completes the proof of the theorem. $\square$

Problem. *Classify spatial embeddings of D_3 up to delta vertex-homotopy or edge-homotopy.*

Remark. We remark here that the α-invariant of a spatial embedding f of K_4 can be interpreted as Milnor's μ-invariant [10] of an associated 3-component link of f.

5. Delta edge-homotopy on θ-curves

By Theorem 2.3, disjoint cycles and a θ-curve are the 'smallest' graphs admitting non-trivial spatial embeddings up to delta edge-homotopy. Then again it is natural to ask the following.

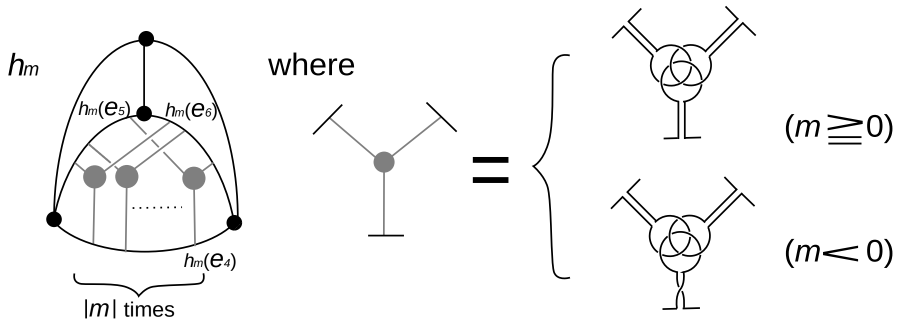

Fig. 4.4

Question. *Can we classify spatial embeddings of disjoint cycles and a θ-curve up to delta edge-homotopy?*

In the case of disjoint cycles, as we noted in Remark just after Definition 1.1, the question above is equivalent to the classification problem of 2-component links up to delta link-homotopy. This question was completely answered as follows.

Theorem 5.1 ([13, 14, 15]). *Let $L = J_1 \cup J_2$ and $M = K_1 \cup K_2$ be oriented 2-component links. Then $L \overset{(\mathrm{DEH})}{\sim} M$ if and only if*

(i) $\mathrm{lk}(L) = \mathrm{lk}(M)$ *and*
(ii) $a_3(L) - \mathrm{lk}(L)\{a_2(J_1) + a_2(J_2)\} = a_3(M) - \mathrm{lk}(M)\{a_2(K_1) + a_2(K_2)\}$.

As the result, it suffice to consider the case of a θ-curve. We prepare some terminology. We give labels to vertices, edges and cycles of a θ-curve, and give orientations to the edges as illustrated below. Given a spatial embedding f of a θ-curve, it is known [7] that there uniquely exists an orientable surface S_f such that S_f has the image of f as a spine and its Seifert linking form vanishes (i.e., all pairwise linking numbers of boundary curves are zero). Then we define the *associated 3-component link* L_f as the boundary $\partial S_f = K_f^1 \cup K_f^2 \cup K_f^3$, see Fig. 5.1.

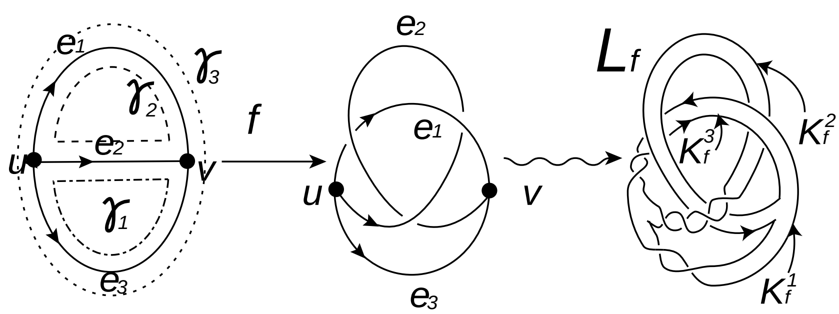

Fig. 5.1. Associated 3-component link

Note that each knot K_f^i is freely homotopic to $f(\gamma_i)$ for $i = 1, 2, 3$. Let $l_1(f) := K_f^2 \cup K_f^3$, $l_2(f) := K_f^3 \cup K_f^1$, $l_3(f) := K_f^1 \cup K_f^2$ be the 2-component sublinks of L_f. Note that these are all algebraically split links. Concerning these links the following is known.

Proposition 5.2 ([31],[4]). *For every spatial embedding f of a θ-curve,*

$$a_3(l_1(f)) = a_3(l_2(f)) = a_3(l_3(f))$$

holds.

Remark. Since the link $l_i(f)$ is algebraically split, we have that the *Sato-Levine invariant* [28] of $l_i(f)$ coincides with $a_3(l_i(f))$ [1].

By virtue of the proposition above, we can define $a_3(f)$ as $a_3(l_i(f))$ for some, and hence, any i. Now we can state our classification theorem.

Theorem 5.3. *Let f and g be two spatial embeddings of a θ-curve. Then $f \overset{(\mathrm{DEH})}{\sim} g$ if and only if $a_3(f) = a_3(g)$.*

Remark. By the result in [34], we can check that

$$a_3(f) \equiv \sum_{i=1}^{3} a_2(f(\gamma_i)) \equiv \widetilde{\alpha}(f) \pmod 2,$$

where the invariant $\widetilde{\alpha}(f)$ was defined in the first example in Section 3.

Example (Kinoshita's θ-curve). Let f be the spatial embedding as illustrated in Fig. 5.2, called *Kinoshita's θ-curve.* This is an example of the almost unknotted theta curve.

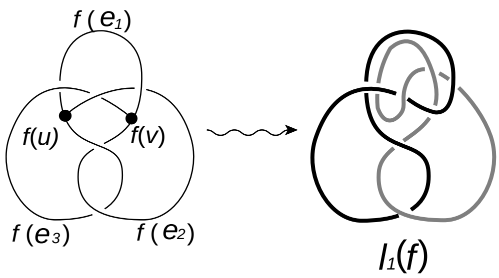

Fig. 5.2

For this f, $a_3(f) = a_3(l_1(f)) = 2$. Thus $f \overset{(\mathrm{DEH})}{\not\sim} h$, where h is the trivial embedding. Remark that the image of f does not contain nontrivial knots, and so, $\widetilde{\alpha}(f) = 0$.

Outline of the proof of Theorem 5.3. The 'only if' part follows from the fact that a_3 is invariant under self-delta moves on 2-component links.

To prove the 'if' part, we again construct a complete system of the representative for the delta edge-homotopy types of spatial embeddings of a θ-curves. We prepare spatial embeddings h_{m,ε_f}, where $m \in \mathbb{Z}$ and $\varepsilon_f \in \{0, 1\}$, as illustrated in Fig. 5.3. By a calculation we can see that $a_3(h_{m,\varepsilon_f}) = 2m + \varepsilon_f$.

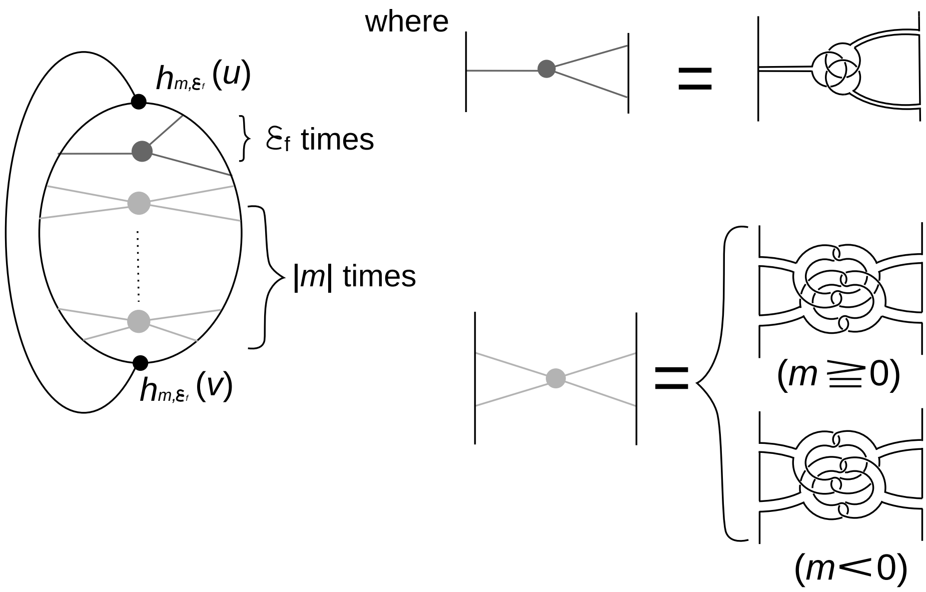

Fig. 5.3

Then we can show that there exists an integer m (resp. n) and $\varepsilon_f \in \{0,1\}$ (resp. $\varepsilon_g \in \{0,1\}$) such that f (resp. g) is delta edge-homotopic to h_{m,ε_f} (resp. h_{n,ε_g}). Thus by the assumption we have that

$$2m + \varepsilon_f = a_3(h_{m,\varepsilon_f}) = a_3(f) = a_3(g) = a_3(h_{n,\varepsilon_g}) = 2n + \varepsilon_g.$$

This implies that $\varepsilon_f = \varepsilon_g$ and $m = n$. Therefore we have that f and g are delta edge-homotopic. This completes the proof of the theorem. □

Problem. *Classify spatial embeddings of K_4 up to delta edge-homotopy.*

Finally we give some corollaries of Theorem 5.3 and discuss about related topics.

Corollary 5.4. *Any boundary θ-curve is delta edge-homotopically trivial.*

Here a spatial embedding f of a θ-curve θ is called a *boundary θ-curve* [25] if there exist compact, connected and orientable surfaces S_1, S_2 and S_3 in S^3 such that $S_i \cap f(\theta) = \partial S_i = f(\gamma_i)$ $(i = 1, 2, 3)$ and $\mathrm{int}S_i \cap \mathrm{int}S_j = \emptyset$ $(i \neq j)$.

Proof. For a boundary theta curve f, any 2-component sublink of L_f must be a boundary link. Since the Conway polynomial of any boundary link is zero [2], we have that $a_3(f) = 0$. Thus we have the result by Theorem 5.3. □

As we noted in Remark just after Definition 1.1, it is known that any 2-component boundary link is delta edge-homotopically trivial. Thus the corollary above is the θ-curve version of this fact. We also note that the converse of the corollary above does not hold. Consider the spatial embedding as illustrated in Fig. 5.5. This is actually delta edge-homotopically trivial, but is not a boundary θ-curve [27].

Besides we mention the relationship between cobordism and delta edge-homotopy on spatial embeddings of a θ-curve. If two spatial embeddings f and g of a θ-curve

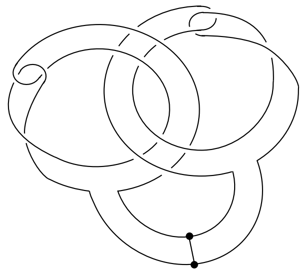

Fig. 5.4. Boundary θ-curve

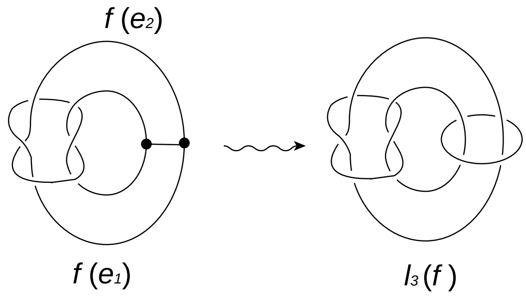

Fig. 5.5

are cobordant, then $l_i(f)$ and $l_i(g)$ are cobordant [31]. Thus we have the following by Theorem 5.3 and the cobordism invariance of the Sato-Levine invariant.

Corollary 5.5. *Let f, g be two spatial embeddings of a θ-curve. Then $f \overset{\text{(Cob)}}{\sim} g$ implies $f \overset{\text{(DEH)}}{\sim} g$. In particular, any slice embedding (recall Definition 0.3(2)) of a θ-curve is delta edge-homotopically trivial.*

It is known that such a implication does not exist for θ_n of $n \geq 4$.

The set of cobordism classes of spatial embeddings of a θ-curve admits a non-abelian group structure with the vertex connected sum [31, 11]. The corollary above suggests that the set of delta edge-homotopy classes, which we denote by $\mathrm{DEH}(\theta)$, of spatial embeddings of a θ-curve might have a group structure.

This is actually true: $\mathrm{DEH}(\theta)$ admits an abelian group structure with the vertex connected sum. Besides we have the following.

Theorem 5.6. *The map $a_3 : \mathrm{DEH}(\theta) \overset{\cong}{\to} \mathbb{Z}$ yields an isomorphism. A generator of $\mathrm{DEH}(\theta)$ is given by the theta curve f as in Figure 3.4.*

Corollary 5.7. *The set of delta edge-homotopy classes represented by almost trivial θ-curves is a subgroup of $\mathrm{DEH}(\theta)$ isomorphic to $2\mathbb{Z}$ under a_3. A generator of $\mathrm{DEH}(\theta)$ is given by Kinoshita's theta curve.*

There are infinitely many spatial embeddings of a θ-curve up to cobordism which are almost trivial and delta edge-homotopically trivial. In fact the set of cobordism classes of almost trivial and delta edge-homotopically trivial spatial embeddings form a subgroup of the θ-curve cobordism group that contains $\mathbb{Z}^\infty$.

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