ON THE CONFIGURATION SPACE OF POINTS AND THE CASSON INVARIANT
(点の配置空間とキャッソン不変量について)

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Abstract. 三次元ホモロジー球面の点の配置空間と接束の自明化を用いてある４次元多様体を構成します．主定理は，その符号数がキャッソン不変量に比例する， というものです．前半はおおまかな構成を述べ， 後半は結果の証明の解説をします．
In this talk, we will construct certain 4-manifold X by using the configuration space of points of an oriented closed homology 3-sphere M and a trivialization of TM. The main theorem is that the signature of X is equal to the Casson invariant of M (up to multiplication by a constant).

Organization

This note was made form the resume written by the speaker together with certain additions and modifications based on the note taken by K. Ichihara. Added or largely modified parts are Section 2, 3, 6, and Subsection 4.1.

1. Introduction

Let M be an oriented closed homology 3-sphere, and $\lambda(M)$ Casson invariant of M. In this report, we construct some topological invariant $I(M)$ such that

- $I(M) = \lambda(M)$ (Theorem 1).
- $I(M) = -\frac{\operatorname{Sign} X_{f_M}}{8}$, where X_{f_M} is a certain 4-dimensional submanifold embedded in the two point configuration space of $M \setminus \{p\}$ (Theorem 2)

And we will also see outline of the proofs of these theorems (§ 8).

Roughly speaking, $\lambda(M)$ is defined by

$$\lambda(M) = \frac{1}{2} \# \frac{\operatorname{Hom}(\pi_1(M), SU(2))^{\mathrm{irr}}}{\text{conjugacy}},$$

and it is known that $\lambda(M)$ is determined by the Dehn surgery formula (c.f. [1]). On the other hand, $\lambda(M)$ is the only one non-trivial invariant which is finite type of degree 1 for both the algebraically split link surgery and Torelli surgery.

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Let X be an oriented compact smooth spin 4-manifold with boundary $\partial X = M$, such X always exists. Rohlin invariant $\mu(M)$ of M is defined by

$$\mu(M) \equiv \frac{\operatorname{Sign} X}{8} \pmod 2,$$

and there is a formula

$$\lambda(M) \equiv \mu(M) \pmod 2.$$

Namely, $\lambda(M)$ is an integral lift of Rohlin invariant $\mu(M)$.

On the face of things, the definitions of Casson invariant and Rohlin invariant looks very different: one is come from the flat $SU(2)$ connections, and the another one is from 4-dimensional. But some relations between Casson invariant and the signature of 4-manifolds are known. Here are such two examples as follows. Let $V(a_1, \ldots, a_n)$ be the Milnor fiber of the Seifert homology 3-sphere $\Sigma(a_1, \ldots, a_n)$, then

$$-\frac{\operatorname{Sign} V(a_1, \ldots, a_n)}{8} = \lambda(\Sigma(a_1, \ldots, a_n))$$

([7],[8]). The another example is about Ohta's invariant $\tau(M)$ (preprint). He constructed some oriented compact smooth 4-manifold N with boundary $\partial N \cong M$ in the moduli space of anti-self dual connections on some principal $SU(2)$ bundle $P \to M \times S^1$, and proved that N is spin and

$$\tau(M) = -\frac{\operatorname{Sign} N}{8}$$

is an topological invariant of M. Hence, one can see that

$$\tau(M) \equiv \mu(M) \pmod 2.$$

It is unknown if $\tau(M) = \lambda(M)$.

As mentioned above, this report gives some topological construction of Casson invariant by using the configuration spaces of 3-manifolds. This results depends on the work by Kuperberg-Thurston [11] that relate our invariant to Casson invariant. Our construction corresponds to the first non-trivial term of their invariant. In [11], they gave a purely topological definition of the perturbative quantum invariants of links and 3-manifolds. Ordinarily, this kind of work for the definition of the perturbative quantum invariants of 3-manifolds and links is by Kontsevich[10]. The related works, which uses the configuration spaces, was given by Axelrod-Singer [2, 3], Bott-Taubes [6], later by Bott-Cattaneo [4, 5], and Kuperberg-Thurston.

2. Basic definitions

In this section we recall basic definitions and notations. Please refer ... for these standard matters.

2.1. **Spin structure.** We start with the definition of the signature of 4-manifolds.

Let X^4 be a closed orientable 4-manifold and $Q : H^2(X,\mathbb{Z})/_{\mathrm{Tor}} \times H^2(X,\mathbb{Z})/_{\mathrm{Tor}} \to \mathbb{Z}$ the intersection form. Then Q is a symmetric, nondegenerate unimodular bilinear form. Set b_+ (resp. b_-) be the number of positive (resp. negative) eugenvalues of Q.

Definition 2.1 (the signature of X)**.** $\operatorname{Sign} X := b_+ - b_-$

Example 2.2.

- $X = \mathbb{C}P^2$. Then $Q = 1$ (i.e. $x \cdot x = 1$ for $\forall x$), and so, $\operatorname{Sign}(Q) = \operatorname{Sign}(1) = 1$.
- $X = S^2_1 \times S^2_2$. Then $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and so, $\operatorname{Sign}(S^2_1 \times S^2_2) = 1$.
- $X = K^3$ surface (or Kummer surface). Let

$$E_8 = \begin{pmatrix} -2 & & & 1 & & & & \\ & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & & \\ 1 & & 1 & -2 & 1 & & & \\ & & & 1 & -2 & 1 & & \\ & & & & 1 & -2 & 1 & \\ & & & & & 1 & -2 & \\ & & & & & & 1 & -2 \end{pmatrix}$$

and then, $Q = E_8 \oplus E_8 \oplus 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Thus $\operatorname{Sign}(X) = b_+ - b_- = -6$.

Concerning the signature of 4-manifolds, one of the most famous theorem is:

Theorem (Rohlin)**.** *If X is smooth and spin, then* $\operatorname{Sign}(X) \equiv 0 \mod 16$.

A spin 4-manifold is defined as follows.

Let $\pi : E \to X$ be a vector bundle over a smooth manifold X and $\{U_\lambda\}$ an open covering of X; $X = \bigcup_\lambda U_\lambda$. By definition of a vector bundle, $E \supset \pi^{-1}(U_\lambda) \cong U_\lambda \times \mathbb{R}^n$ (n denotes the dimension of the fiber) and there exists $g_{UV} : U \cap V \to GL(n,\mathbb{R})$ satisfying $g_{UV} g_{VW} g_{WV} = 1$ for $U \cap V \cap W \neq \emptyset$ (called the *cocycle condition*). Up to homotopy we may assume that each $g_{UV} \in SO(n) \subset GL(n,\mathbb{R})$.

Recall that $\pi_1(SO(n)) \cong \mathbb{Z}_2$ for $n \geq 3$. We define $Spin(n)$ by the double covering $\pi : Spin(n) \overset{\mathbb{Z}_2}{\to} SO(n)$.

Definition 2.3 (spin structure on a vector bundle E, (I))**.** A *spin structure* σ on E is defined as $\sigma = \{\widetilde{g_{UV}}\}$ such that $\widetilde{g_{UV}} : U \cap V \to Spin(n)$ with $\pi(\widetilde{g_{UV}}) = g_{UV}$ and $\widetilde{g_{UV}}\widetilde{g_{VW}}\widetilde{g_{WV}} = 1 \in Spin(n)$.

Definition 2.4 (spin structure on a vector bundle E, (II))**.** A *spin structure* σ on E is defined as a spin structure σ on $E \oplus \mathbb{R}^N$ for some $N \geq 2$.

It is known that these two definitions make no contradiction.

Definition 2.5. A smooth manifold X is *spin* if its tangent bundle admits a spin structure.

Here we give some remarks. Given $g_{UV} : U \cap V \to GL(n, \mathbb{R})$ and $\pi_1(SO(n)) \cong \mathbb{Z}_2$, a local lift $\widetilde{g_{UV}} : U \cap V \to Spin(n)$ always exists. For such local lifts, set $h_{UVW} := \widetilde{g_{UV}}\widetilde{g_{VW}}\widetilde{g_{WV}}$. Then $\{h_{UVW}\}$ gives a cocycle in $C^2(X, \mathcal{U}, \mathbb{Z}_2)$ with some open covering $\mathcal{U}$ of X. This cocycle represents the cohomology class in $H^2(X, \mathcal{U}, \mathbb{Z}_2)$, which is equal to the Stiefel-Whitney class w_2. Thus we have the following:

Fact. *On a vector bundle $\pi : E \to X$ over a smooth manifold X, there exists a spin structure if and only if the Stiefel-Whitney class $w_2 \in H^2(X, \mathbb{Z}_2)$ is zero.*

From this fact, we can observe that a spin structure σ on a vector bundle E over X gives a 'trivialization' next to the 'orientation'. That is; by using σ, one can have $E|_{X^{(2)}} \cong X^{(2)} \times \mathbb{R}^n$, where $X^{(2)}$ denotes the 2-skeleton of X endowed with a CW-complex structure.

2.2. **Rohlin invariant.** In this subsection we introduce Rohlin invariant of 3-manifolds.

Let M^3 be an oriented closed homology 3-sphere (i.e., $H_*(M, \mathbb{Z}) = H_*(S^3, \mathbb{Z})$). Note that

- M is spin, for $w_2 \in H^2(TM, \mathbb{Z}_2) = 0$, and
- the spin structure on TM is unique up to homotopy, for the difference $diff(\sigma_1, \sigma_2)$ of the spin structures σ_1, σ_2 on TM lies in $H^1(X, \mathbb{Z}_2)$, which actually vanishes for a homology 3-sphere M.

Moreover we have the following facts.

Fact.

(1) *There exists an oriented compact smooth spin simply-connected 4-manifold X such that $\partial X = M$ (originally due to Thom).*

(2) *For such X, the intersection form $Q : H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to \mathbb{Z}$ is non-degenerate, and $\mathrm{Sign}(Q) \equiv 0 \mod 8$, i.e., $\frac{\mathrm{Sign}(Q)}{8} \in \mathbb{Z}$. This follows from the facts that M is spin and an algebraic property of Q: Q is an even form.*

Now we define:

Definition 2.6 (Rohlin invariant $\mu(M)$ of M)**.** $\mu(M) := \frac{\mathrm{Sign}(X)}{8} \in \mathbb{Z}_2$

This $\mu(M)$ is well-defined: It suffice to show that it is independent from the choice of the 4-manifold with boundary M. Let X' be another 4-manifold with $\partial X' = M$. By the Novikov additivity, $\operatorname{Sign}((-X') \cup X) = -\operatorname{Sign}(X') + \operatorname{Sign}(X)$ holds, where $-X'$ denotes a copy of X' with opposite orientation. This $\operatorname{Sign}((-X') \cup X)$ have to be zero modulo 16 by the Rohlin's theorem, since $(-X') \cup X$ is a closed smooth spin 4-manifold. It concludes that $\operatorname{Sign}(X') \equiv \operatorname{Sign}(X) \mod 16$, and so, $\frac{\operatorname{Sign}(X)}{8} = \frac{\operatorname{Sign}(X')}{8} \in \mathbb{Z}_2$.

2.3. **Casson invariant.** In this subsection we give some facts and a conjecture about Casson invariant, which is a motivation of my work.

As in Section 1, the original definition (due to Casson) of Casson invariant $\lambda(M)$ is given, very roughly, by

$$\lambda(M) = \frac{1}{2} \# \frac{\operatorname{Hom}(\pi_1(M), SU(2))^{\mathrm{irr}}}{\text{conjugacy}}.$$

This is shown to be an integer-valued topological invariant for oriented closed homology 3-spheres.

In present, another definition is also known. This is axiomatical; $\lambda(M)$ is determined by the following inductively.

- $\lambda(S^3) = 0$.
- $\lambda(-M) = \lambda(M)$.
- $\lambda(M_1 \# M_2) = \lambda(M_1) + \lambda(M_2)$ for the connected sum $M_1 \# M_2$.
- $\lambda(M_K) - \lambda(M) = \frac{1}{2}\Delta''_K(1)$ (Dehn surgery formula).

See [1] for example.

As we stated in Section 1, Casson invariant $\lambda(M)$ can be regarded as an integral lift of Rohlin invariant $\mu(M)$: that is, a formula

$$\lambda(M) \equiv \mu(M) \pmod 2$$

is known.

Since Rohlin invariant is defined as $\mu(M) \equiv \frac{1}{8}\operatorname{Sign}(X) \mod 2$ for some 4-manifold X with boundary M, it might be possible that one can find a 4-manifold X with boundary M such that $\lambda(M) = \frac{1}{8}\mu(M)$. Concerning this observation, the following conjecture is known.

Conjecture (Casson Invariant Conjecture (Neumann))**.** *Signature of* (*some special*) *Milnor fiber of* M *is equal to* $8\lambda(M)$.

See *** for detail about this conjecture. Some partial positive answers to the conjecture have been obtained.

Theorem (Fintushel and Stern [7]). *Consider* $f(X,Y,Z) = X^p + Y^q + Z^r$ *with* p,q,r *coprime integers. Let* $V(f) := \{(X,Y,Z) \in \mathbb{C}^3 \mid f(X,Y,Z) = 0\} \subset \mathbb{C}^3$ *and* $\Sigma(p,q,r) := V(f) \cap S^5$. *Here we regard* S^5 *as the unit sphere in* $\mathbb{R}^6 = \mathbb{C}^3$. *It is shown that the map*

$$\frac{f}{|f|} : S^5 \setminus \Sigma(p,q,r) \to S^1$$

gives a fiber bundle structure. After suitable compactification, the fiber (called Milnor fiber) is regarded as a 4*-manifold* X *with* $\partial X = \Sigma(p,q,r)$. *Then*

$$\lambda(\Sigma(p,q,r)) = \frac{\operatorname{Sign} X}{8}$$

holds.

In other words, for $\Sigma(p,q,r)$, the signature of a Milnor fiber X is a topological invariant which is an integral lift of Rohlin invariant.

3. Idea

What we see in this section is the background idea to get an invariant which is an integral lift of Rohlin invariant. To do this, we use the *two point configuration space* of a 3-manifold. See the next section for precise definitions of the terms in the following.

3.1. **Finding X_f^4.** Let (M,f) be a 3-manifold M and a framing f of the 'punctured' M. Consider the two point configuration space

$$C_2(\hat{M}) := M \times M \setminus (* \times M \cup M \times * \cup \{(x,x)\})$$

of 'punctured' M, which is assumed to be compactified 'suitably'. Thus this is a compact 6-manifold with non-empty boundary.

We will construct a 'partial Gauss map' $\varphi_f \colon U \to S^2$, where U is a complement of a certain compact subset of $C_2(\hat{M})$ defined by using f. Intuitively this map means

$$M \times M \ni (x,y) \mapsto \frac{y-x}{||y-x||} \in S^2.$$

By the way of compactification, this map naturally extends to the map of $\partial C_2(\hat{M})$.

Now we suppose the existence of a map

$$\tilde{\varphi}_f \colon C_2(\hat{M}) \to S^2$$

such that $\tilde{\varphi}_f|_{\partial C_2(\hat{M})} = \varphi_f$. Taking a regular value $v \in S^2$ of $\tilde{\varphi}_f$, set

$$X_f = \tilde{\varphi}_f^{-1}(v).$$

Then X_f is an oriented compact smooth 4-manifold satisfying

(1) $\partial X_f \cong M \# M \# (-M)$,

(2) X_f is spin,

(3) the signature σ_f of X_f depends only on (M, f).

The property (1) is from the definition of φ_f. Since the normal bundle of $X_f \subset C_2(\hat{M})$ and the tangent bundle $TC_2(\hat{M})$ are spin, X_f is also spin, so we have (2). Another manifold $X_f{}'$ come from another map $\tilde{\varphi}'_f : C_2(\hat{M}) \to S^2$ is cobordant to X_f relative to the boundary, hence. Consequently, σ_f is an invariant of a pair (M, f), and we obtain (3). By the definition of Rohlin invariant, one can see

$$\frac{\sigma_f}{8} \equiv \mu(M) \pmod 2.$$

Therefore, we get an integral lift $\sigma_f/8$ of Rohlin invariant.

3.2. **Finding $\tilde{\varphi}_f$.** In this subsection we explain an idea to find $\tilde{\varphi}_f : C_2(\hat{M}) \to S^2$ such that $\tilde{\varphi}_f|_{\partial C_2(\hat{M})} = \varphi_f$.

Suppose that φ_f as before is already given. We want to apply 'obstruction theory'.

Lemma. *we have*

$$H^i(C_2(\hat{M}), \partial C_2(\hat{M})) \cong H^i(M \times M, A) = \begin{cases} \mathbb{Z} & i = 4, 6 \\ 0 & \textit{otherwise} \end{cases}$$

where $A = * \times M \cup M \times * \cup \{(x, x)\}$.

This follows from

$$H^i(M \times M) = \begin{cases} \mathbb{Z} & i = 1 \\ \mathbb{Z}^2 & i = 4, 6 \\ 0 & \text{otherwise} \end{cases}$$

and

$$H^i(A) = \begin{cases} \mathbb{Z} & i = 1 \\ \mathbb{Z}^3 & i = 4 \\ 0 & \text{otherwise} \end{cases}.$$

By the lemma above the primary obstruction class $o^4(\varphi_f)$ lies in

$$H^4(C_2(\hat{M}), \partial C_2(\hat{M}); \pi_3 S^2) \cong H^4(C_2(\hat{M}), \partial C_2(\hat{M}); \mathbb{Z}) \cong \mathbb{Z}.$$

If we can establish $o^4(\varphi_f) = 0$, i.e., we obtain $\varphi_f^{(4)} : X^{(4)} \to S^2$ such that $\varphi_f^{(4)}|_{\partial C_2(\hat{M})} = \varphi_f$, where $X^{(4)}$ denotes the 4-skeleton of $(C_2(\hat{M}), \partial C_2(\hat{M}))$, the secondary obstruction class $o^6(\varphi_f^{(4)})$ lies in

$$H^4(C_2(\hat{M}), \partial C_2(\hat{M}); \pi_5 S^2) \cong H^4(C_2(\hat{M}), \partial C_2(\hat{M}); \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

We will in fact achieve $o^4(\varphi_f) = 0$. However, unfortunately, we cannot get rid of $o^6(\varphi_f^{(4)})$, and so, we will take another way to construct our invariant.

4. Configuration space of points and Gauss map

In this section we first introduce the two point configuration space of a 'punctured' 3-manifold and compactify it. Then we define a 'partial Gauss map' φ_f as explained in the previous section.

4.1. $C_2(\hat{M})$ and $\partial C_2(\hat{M})$.

The *two point configuration space* $C_2(X)$ of a space X is defined by

$$C_2(X) = \{(x, y) \mid x, y \in X,\ x \neq y\}.$$

Let M be an oriented closed homology 3-sphere and set $\hat{M} = M \# \mathbb{R}^3$. We compactify $C_2(\hat{M})$ by the method of Bott-Taubes's paper.

Let $\Delta^3 := \{(x, x, x)\} \subset M \times M \times M$. This can be regarded as an image of a smooth embedding of M into $M \times M \times M$. Also let $\Delta^2 := \{(x, x, y), (x, y, x), (y, x, x)\} \subset M \times M \times M$. Obviously Δ^3 is a subset of Δ^2. Thus $\Delta^2 \cap (M \times M \times M \setminus N_3)$ is smoothly embedded into $M \times M \times M \setminus N_3$, where N_3 denotes an open neighborhood of Δ^3. Let N_2 denote an open neighborhood of $\Delta^2 \cap (M \times M \times M \setminus N_3)$. Set $C_3^c(M) := \times M \times M \setminus N_2 \setminus N_3$, which in fact gives a compactification of $C_3(M)$.

Now we define a compactification $C_2^c(\hat{M})$ of $C_2(\hat{M})$ as the following diagram commutes.

$$\begin{array}{ccc} C_2^c(\hat{M}) & \longrightarrow & C_3^c(M) \ni (x, y, z) \\ \downarrow & & \downarrow \\ \{p\} & \longrightarrow & M \ni z \end{array}$$

This compactification gives a homotopy equivalence between $C_2^c(\hat{M})$ and $C_2(\hat{M})$. In the following we abuse $C_2(\hat{M})$ to also denote $C_2^c(\hat{M})$.

4.2. The map $\varphi_f \colon \partial C_2(\hat{M}) \to S^2$.

In this subsection, we construct a "partial Gauss map" $\varphi_f \colon U \to S^2$, where U is a complement of a certain compact subset of $C_2(\hat{M})$ by using some additional data f. By the definition of the compactification of $C_2(\hat{M})$, φ_f defined on U naturally extends to $\partial C_2(\hat{M})$, which gives a fiber bundle structure of $\partial C_2(\hat{M})$ over S^2.

At first we give φ_f for the Euclidean space $\mathbb{R}^3$ as an easiest, but instructive example. Let

$$\varphi_{\mathbb{R}^3} \colon C_2(\mathbb{R}^3) \to S^2$$

be the map defined by

$$\varphi_{\mathbb{R}^3}(x, y) = \frac{y - x}{\|y - x\|}$$

for $(x, y) \in C_2(\mathbb{R}^3)$ which is called Gauss map. There exists a homeomorphism

$$C_2(\mathbb{R}^3) \cong \mathbb{R}^3 \times (0, \infty) \times S^2, \quad (x, y) \mapsto (z, r, v)$$

by the corresponding

$$z = \frac{x+y}{2}, \quad r = \frac{\|y-x\|}{2}, \quad v = \varphi_{\mathbb{R}^3}(x,y).$$

Immediately, the pre-image ${\varphi_{\mathbb{R}^3}}^{-1}(v)$ of a point $v \in S^2$ is contractible. In particular, its signature is zero. As we will see later (Theorem 2), this vanishing corresponds to the fact $\lambda(S^3) = 0$.

Let M be an oriented closed homology 3-sphere and set $\hat{M} = M\#\mathbb{R}^3$. Let $f\colon T\hat{M} \to \hat{M} \times \mathbb{R}^3$ be a framing of $T\hat{M}$. In this report, we always assume that any framing of $T\hat{M}$ is compatible with Euclidean framing $f_{\mathbb{R}^3}$ over the complement of some compact subset of $\hat{M}$. Now, we define a map $\varphi_f\colon U \to S^2$ as follows, where $U = U_1 \cup U_2 \cup U_3$. Each U_i is defined in later three cases.

4.2.1. *Two points are very close.* First, let

$$U_1 = \{(x,y) \in C_2(\hat{M}) \mid d(x,y) < \varepsilon\},$$

where d is a metric of $\hat{M}$ and $\varepsilon > 0$ is small enough. Suppose $(x,y) \in U_1$. Then we can define the "direction" $\varphi_f(x,y) \in S^2$ from x to y by using f.

4.2.2. *one point is in the end of $\hat{M}$.* Suppose that $\hat{M}$ is obtained by connecting a 3-disk $D_p \subset M$ around $p \in M$ and $D^3 \subset \mathbb{R}^3$ around 0. Let $V \subset \hat{M}$ be the open subset corresponding to $M \setminus D_p$, and $W \subset \hat{M}$ corresponding $\mathbb{R}^3 \setminus D^3$. Let

$$U_2 = (V \times W) \cup (W \cup V).$$

We assume that f is coincides with $f_{\mathbb{R}^3}$ on some open set including the closure of W.

If $x \in V$ and $y \in W$, then define $\varphi_f(x,y) = y$. Note that this definition makes sense, because we can think y as a point in $\mathbb{R}^3$. Also define $\varphi_f(y,x) = -y$.

4.2.3. *Both two points in the end.* Let

$$U_3 = C_2(W),$$

and suppose $(x,y) \in U_3$. In this case, we define

$$\varphi_f(x,y) = \frac{y-x}{\|y-x\|}.$$

From the way of compactification, we have the following.

Lemma. *The map $\varphi_f\colon U \to S^2$ naturally extends to $\varphi_f\colon \partial C_2(\hat{M}) \to S^2$* (we abuse the notation φ_f). *The fiber $\varphi_f^{-1}(p)$ for every regular value $p \in S^2$ of $\varphi_f\colon \partial C_2(\hat{M}) \to S^2$ is diffeomorphic to $M\#M\#(-M)$.*

5. Two numbers d_f and σ_f

In this section, we define two integers d_f and σ_f used to define the invariant $I(M)$.

Let

$$\bar{C}_2(\hat{M}, f) = S^2 \cup_{\varphi_f} C_2(\hat{M})$$

be the attaching space by the map $\varphi_f \colon \partial C_2(\hat{M}) \to S^2$. Using the long exact sequence of $(\bar{C}_2(\hat{M}, f), S^2)$, we have an isomorphism

$$H^k(\bar{C}_2(\hat{M}, f); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, 6 \\ 0, & \text{otherwise.} \end{cases}$$

The following lemma define the integer d_f, this definition is similar to the cohomological definition of the Hopf invariant(c.f. [9]).

Lemma 5.1. *There exists a graded ring isomorphism*

$$H^*(\bar{C}_2(\hat{M}, f); \mathbb{Z}) \cong \mathbb{Z}[a, b]/(a^2 - d_f b, b^2)$$

for some integer d_f, where $\deg a = 2$, $\deg b = 4$.

This number d_f is nothing but the Casson invariant for the framed 3-manifold ([11]).

Let $L_{S^2} \to S^2$ be a complex line bundle with Euler number 1, and s_{S^2} a generic section. We can assume that ${s_{S^2}}^{-1}(0)$ consists just one point. Set $L_f = {\varphi_f}^* L_{S^2}$, $s_f = {\varphi_f}^* s_{S^2}$. Then $s_f : \partial C_2(\hat{M}) \to L_f$ is a generic section, and ${s_f}^{-1}(0) \cong M \# M \# (-M)$. Since the inclusion $\partial C_2(\hat{M}) \hookrightarrow C_2(\hat{M})$ induces an isomorphism on H^2, there exists only one isomorphism class of a complex line bundle $\tilde{L}_f \to C_2(\hat{M})$ such that $\tilde{L}_f|_{\partial C_2(\hat{M})} = L_f$.

$$\begin{array}{ccc} L_f & \longrightarrow & \tilde{L}_f \\ \downarrow & & \downarrow \\ \partial C_2(\hat{M}) & \longrightarrow & C_2(\hat{M}) \end{array}$$

Let $\tilde{s}_f : C_2(\hat{M}) \to \tilde{L}_f$ be a generic section such that $\tilde{s}_f|_{\partial C_2(\hat{M})} = s_f$. Let

$$X_f = \tilde{s}_f^{-1}(0),$$

then X_f is an oriented compact smooth 4-manifold with boundary $\partial X_f \cong M \# M \# (-M)$. Define

$$\sigma_f = \operatorname{Sign} X_f.$$

Another choices $\tilde{s}'_f$ give the same value of σ_f, because a generic homotopy between $\tilde{s}_f$ and $\tilde{s}'_f$ gives a cobordism between X_f and X'_f.

Definition 5.2.

$$I(M) = \frac{d_f - \sigma_f}{8}$$
$$I(M, f) = (I(M), d_f)$$

6. Legitimacy of invariants

We have just defined our 'invariant' $I(M)$, which will be shown to be an integer-valued, topological invariant of M (Section 8) and to be equal to Casson invariant of M (Section 9).

In this section, we try to give another 'definition' of $I(M, f)$, which could explain the legitimacy of $I(M, f)$. The idea behind the explanation was given in Subsection 3.2. Please remark that this section contains some unreliable arguments.

Now we have the map $\varphi_f : \partial C_2(\hat{M}) \to S^2$ and the attaching space $\bar{C}_2(\hat{M}, f) = S^2 \cup_{\varphi_f} C_2(\hat{M})$ by φ_f. This $\bar{C}_2(\hat{M}, f)$ is smooth except on S^2. In fact, the 'normal disk' at $p \in \bar{C}_2(\hat{M}, f) \setminus S^2$ is regarded as a cone over a generic fiber $\varphi_f^{-1}(p) \cong M\#M\#(-M)$. Here we assume that:

Assumption. *The attaching space $\bar{C}_2(\hat{M}, f)$ is smooth everywhere.*

Let $\bar{L}_f \to \bar{C}_2(\hat{M}, f)$ be the complex line bundle naturally obtained from $\tilde{L}_f \to C_2(\hat{M})$ and $L_{S^2} \to S^2$, and then, consider the spin bordism class of $\bar{L}_f \to \bar{C}_2(\hat{M}, f)$. The *n-dimensional spin bordism group* is defined by

$$\Omega_n^{spin}(X) := \left\{ (Z, h) \;\middle|\; \begin{array}{l} Z: \text{ spin closed smooth } n\text{-manifold} \\ h: Z \to X, \text{ continuous map} \end{array} \right\} \Big/ \text{cobordant}$$

Here (Z_1, h_1) is said to be *cobordant* to (Z_2, h_2) if there exists a spin $(n+1)$-manifold W and a continuous map $\tilde{h}: W \to X$ such that $\partial W = Z_1 \amalg (-Z_2)$ and $\tilde{h}|_{\partial W} = h_1 \amalg h_2$.

Example 6.1. Define $\Omega_n^{spin} := \Omega_n^{spin}(*)$. Then the following table is known.

n	0	1	2	3	4	5	6	7
Ω_n^{spin}	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0

In detail:

- The generator of $\Omega_1^{spin} \cong \mathbb{Z}_2$ given by Lie framing, i.e., the framing obtained from a framing at one point by distributing by the elements of the Lie group S^1.
- $\Omega_3^{spin} = 0$ indicates that there always exists an oriented compact spin 4-manifold bounded by given closed 3-manifold.
- The isomorphism $\Omega_4^{spin} \cong \mathbb{Z}$ is given by the correspondence $X^4 \mapsto \frac{\text{Sign}\, X}{16}$.

In the case of $n = 6$ and $X = \mathbb{C}P^\infty$ we have the following.

Proposition. $\Omega_6^{spin}(\mathbb{C}P^\infty) \cong \mathbb{Z} \oplus \mathbb{Z}$

The group $\Omega_n^{spin}(\mathbb{C}P^\infty)$ is called the *n-dimensional spin bordism group decorated with a complex line bundle*, for there exists a correspondence between the complex line bundles over Z and homotopy classes of continuous maps of Z to $\mathbb{C}P^\infty$. Using this correspondence, the isomorphism in the proposition above is given by

$$[L \to Z] \mapsto \left(\frac{d-\sigma}{8}, d\right)$$

where Z denotes a 6-manifold, L a complex line bundle over Z, $d := \int_Z c_1(L)^3$ with $c_1(L)$ the first Chern form, and $\sigma := \operatorname{Sign} s^{-1}(0)$ with $s\colon Z \to L$ a generic section.

In our setting $Z = \bar{C}_2(\hat{M}, f)$ and $L = \bar{L}_f$, this value $\left(\frac{d-\sigma}{8}, d\right)$ is actually coincident with $I(M, f) = (\frac{d_f - \sigma_f}{8}, d_f)$.

7. Statements and Examples

Let (M, f) be an oriented closed homology 3-sphere with a framing of $T\hat{M}$. Let $I(M)$ be the number defined in Definition 5.2.

Theorem 1 ([14])**.** *$I(M)$ is a $\mathbb{Z}$-valued topological invariant of M, and it equals to Casson invariant of M.*

Now, we see some examples of the calculations of $I(M)$ and $I(M, f)$.

7.1. **Case of $M = S^3$ with Euclidean framing.** Let $M = S^3$, then $\hat{M} = \mathbb{R}^3$. Let $f_{\mathbb{R}^3}$ be the Euclidean framing on $T\mathbb{R}^3$, and we have $\varphi_{f_{\mathbb{R}^3}} = \varphi_{\mathbb{R}^3}$.

One can take the line bundle $\tilde{L}_{f_{\mathbb{R}^3}} \to C_2(\hat{M})$ as the pull-back bundle ${\varphi_{f_{\mathbb{R}^3}}}^* L_{S^2}$, and the pull-back section

$$\tilde{s}_{f_{\mathbb{R}^3}} = \varphi_{\mathbb{R}^3}^* s_{S^2} \colon C_2(\hat{M}) \to \tilde{L}_{f_{\mathbb{R}^3}}$$

is generic. If $v = s_{S^2}^{-1}(0)$, then we have

$$X_{f_{\mathbb{R}^3}} = \tilde{s}_{f_{\mathbb{R}^3}}^{-1}(0) = \varphi_{\mathbb{R}^3}^{-1}(v) \cong \mathbb{R}^4,$$

and so, we obtain $\sigma_{f_{\mathbb{R}^3}} = 0$.

Since $\bar{C}_2(\mathbb{R}^3) \cong S^4 \times S^2$, we have the ring isomorphism

$$H^*(\bar{C}_2(\hat{M}, f); \mathbb{Z}) \cong R_0,$$

this implies $d_{f_{\mathbb{R}^3}} = 0$. Therefore, we have

$$I(S^3) = \frac{d_{f_{\mathbb{R}^3}} - \sigma_{f_{\mathbb{R}^3}}}{8} = \frac{0-0}{8} = 0$$

and

$$I(S^3, f_{\mathbb{R}^3}) = (I(M), d_{f_{\mathbb{R}^3}}) = (0, 0).$$

7.2. **Case of $M = S^3$ with any framings.** Let f be an any framing of $\mathbb{R}^3$ which is always obtained by $f = gf_{\mathbb{R}^3}$ for some compact supported map $g\colon \mathbb{R}^3 \to SO(3)$. Any framings on $\mathbb{R}^3$ are classified by the degree $\deg g \in \mathbb{Z}$ of the induced homomorphism $g^*\colon H^3_c(SO(3);\mathbb{Z}) \to H^3_c(\mathbb{R}^3;\mathbb{Z})$.

Let $n = \deg g$. The map $\varphi_f\colon \partial C_2(\hat{M}) \cong S^3 \times S^2 \to S^2$ essentially equals to the evaluation map ev_g as follows:

$$ev_g\colon S^3 \times S^2 \to S^2, \quad (x, v) \mapsto g(x)v$$

Therefore, we have $\bar{C}_2(\hat{M}, f) \cong S(E_n)$, where $\pi\colon E_n \to S^4$ is a real vector bundle with $\langle [S^4], p_1(E_n)\rangle = 4n$ and $S(E_n)$ the associated sphere bundle. Calculating the characteristic classes of $TS(E_n)$ and $\pi^* E_n$([13]), we obtain that $d_f = n$ and $\sigma_f = n$. This implies that

$$I(S^3, f) = (\frac{d_f - \sigma_f}{8}, d_f) = (0, n),$$

and of course, we obtain $I(S^3) = 0$ again.

7.3. **Connected sum.** Let (M_1, f_1), (M_2, f_2) be framed manifolds. Set $M = M_1 \# M_2$, $f = f_1 \# f_2$. Let us think $\hat{M}_1 = M_1 \# \mathbb{R}_1$, $\hat{M}_2 = M_2 \# \mathbb{R}_2$, where

$$\mathbb{R}_1 = \{(x_1, x_2, x_3) \mid x_1 < 0\}, \quad \mathbb{R}_2 = \{(x_1, x_2, x_3) \mid x_1 > 0\},$$

and each f_i is a faming over $\hat{M}_1$ compatible with the Euclidean framing on the end. Moreover, we suppose that $\hat{M}_i$ is the connected sum at around a point in M_i and around $(\pm R, 0, 0) \in \mathbb{R}_i$ for some large number $R \gg 1$. Then we can take $\hat{M}$ as

$$\hat{M}_1 \cup V \cup \hat{M}_2$$

such that the M_1-part and M_2-part in $\hat{M}$ are very far each other, where $V = (-1, 1) \times \mathbb{R}^2$.

Next, we will construct $\varphi_f\colon \partial C_2(\hat{M}) \to S^2$. Define a map

$$\varphi_{ij}\colon (V \cup \hat{M}_i) \times (V \cup \hat{M}_j) \setminus \Delta \to S^2$$

as follows $(i, j = 1, 2,\ i \neq j)$. Let $h\colon \hat{M} \to \mathbb{R}^3$ be the map obtained by collapsing each M_i-part to $(\pm R, 0, 0)$. Then $\varphi_{ij}(x, y)$ is defined by

$$\varphi_{ij}(x, y) = \frac{h(y) - h(x)}{\|h(y) - h(x)\|}.$$

Let $\tilde{\varphi}_{f_i}\colon C_2(M_i) \to \mathbb{C}P^3$ be the classifying map of $\tilde{L}_{f_i}$. Let $\tilde{\varphi}_f\colon C_2(\hat{M}) \to \mathbb{C}P^3$ be an extension map of $\tilde{\varphi}_f$ obtained from $\tilde{\varphi}_{f_1}$, $\tilde{\varphi}_{f_2}$ and φ_{ij}. Note that any two such maps are coincide on these common domain. Let $\mathbb{C}P^{2'} \subset \mathbb{C}P^3$ be a submanifold homologous to $\mathbb{C}P^2$ that transversally intersect with $\mathbb{C}P^1 \cong S^2$ at one point $v = (1,0,0) \in S^2$. There exists a generic section of the complex line bundle $L_{\mathbb{C}P^3} \to \mathbb{C}P^3$ with $c_1(L_{\mathbb{C}P^3}) = 1$ such that the pre-image of zero is $\mathbb{C}P^{2'}$. Hence

$$X_f = \varphi_f(\mathbb{C}P^{2'}) \cong X_{f_1} \sharp X_{f_2},$$

this means $\sigma_f = \sigma_{f_1} + \sigma_{f_2}$. And also it is easy to see that $d_f = d_{f_1} + d_{f_2}$. Therefore, we obtain the following:

Proposition 7.1.

$$I(M,f) = I(M_1,f_1) + I(M_2,f_2)$$
$$I(M) = I(M_1) + I(M_2)$$

7.4. **Opposite orientation.** Let M' be M with the opposite orientation, and let $f' = (-f_1, f_2, f_3)$ where $f = (f_1, f_2, f_3)$. By the definition of d_f and σ_f, we have

$$d_{f'} = -d_f, \quad \sigma_{f'} = -\sigma_f.$$

This implies that

Proposition 7.2.

$$I(M', f') = -I(M,f)$$
$$I(M') = -I(M)$$

8. Outline of proof

8.1. **Integrality of $I(M)$.** Since $\Omega_5^{spin}(S^2) = 0$([15]), there exists an oriented compact smooth spin 6-manifold Z with a complex line bundle $L_Z \to Z$ such that

$$\partial Z = \partial C_2(\hat{M}), \quad L_Z|_{\partial C_2(\hat{M})} = L_f$$

and the image of the classifying map of L_Z is contained in S^2. Let

$$W = C_2(\hat{M}) \cup_{\partial C_2(\hat{M})} Z, \quad L_W = \tilde{L}_f \cup_{L_f} L_Z \to W.$$

Then, (W, L_W) is an oriented closed smooth spin 6-manifold with a complex line bundle. Applying the index theorem to (W, L_W), one can see that the integral

$$\int_W ch(L_W)\hat{\mathcal{A}}(TW)$$

is an integer(c.f. [12]). Here, ch is the Chern character and $\hat{\mathcal{A}}$ is the $\hat{\mathcal{A}}$-genus. This value equals to

$$I(M) - \frac{\operatorname{Sign} X_Z}{8},$$

where X_Z is the pre-image of 0 of a generic section, which is an extension of s_f of L_f, of L_Z. Since X_Z is spin and ∂X_Z is a homology 3-sphere, we have $\operatorname{Sign} X_Z \equiv 0 \pmod 8$. Therefore, we obtain the following proposition.

Proposition 8.1. *The number $I(M)$ is an integer.*

8.2. **Topological invariance of $I(M)$.** Let f, f' be framings of $T\hat{M}$. There exists a one-to-one correspondence between the set of homotopy classes of framings on $\hat{M}$ and $[\hat{M}, SO(3)]_c$, i.e., the set of maps with compact supports. Thus f' can be represented by $f' = gf$ for some $g\colon \hat{M} \to SO(3)$.

Then $(M, f') \cong (M\#S^3, f\#gf_{\mathbb{R}^3})$. According to Proposition 7.1,

$$I(M, f') = I(M, f) + I(S^3, gf_{\mathbb{R}^3}) = I(M, f) + (0, \deg g).$$

In other words,

$$d_{f'} = d_f + \deg g, \quad \sigma_{f'} = \sigma_f + \deg g.$$

In particular, $I(M)$ does not depend on f. Therefore

Proposition 8.2. *$I(M)$ is an topological invariant of M.*

8.3. **Casson invariant.** Kuperberg-Thurston showed that some value $\tilde{I}_1(M)$ is Casson invariant by using the theory of finite type invariants of homology 3-spheres in their paper [11]. The invariant $\tilde{I}(M)$ is constructed as follows. First, define

$$I_1(M) = \frac{1}{6}\langle \bar{C}_2(\hat{M}, f), c_1(L_f)^3 \rangle,$$
$$\delta_1(M) = \frac{1}{24}\langle X_f, p_1(TC_2(\hat{M})|_{X_f}) \rangle.$$

And then $I(M)$ is defined by

$$\tilde{I}_1(M) = I_1(M) - \delta_1(M).$$

Calculating characteristic classes, we have the following proposition.

Proposition 8.3. $\tilde{I}_1(M) = I(M)$.

By Proposition 8.3, we obtain Theorem 1. To prove that $I(M)$ is a $\mathbb{Z}$-valued topological invariant, one only need Proposition 8.1,8.2.

9. Casson invariant as a signature

By § 8.2, we obtain the following proposition.

Proposition 9.1. *There exists only one framing f_M of $T\hat{M}$ such that $d_{f_M} = 0$.*

Therefore, we have

$$I(M) = \frac{d_{f_M} - \sigma_{f_M}}{8} = -\frac{\sigma_{f_M}}{8}.$$

Theorem 2.

$$\lambda(M) = -\frac{\sigma_{f_M}}{8}.$$

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