NOTE FOR

'WORKSHOP OF FLEDGLINGS ON LOW-DIMENSIONAL TOPOLOGY'

# PSEUDO-ANOSOV BRAIDS ON THE 2-SPHERE

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Abstract. A correspondence between braids on the 2-disk and those on the 2-sphere is naturally induced from the inclusion map of the 2-disk into the 2-sphere. A natural necessary condition for a pseudo-Anosov braid on the 2-disk so that the corresponding braid on 2-sphere is also pseudo-Anosov. It is shown that this condition is not sufficient in general.

## 1. Preliminary

The main subject of this talk is the natural correspondence between braids on the 2-disk and those on the 2-sphere. Let $b$ be an $n$-braid on $\mathbf{D^2}$ with $n \geq 3$. A correspondence between braids on $\mathbf{D^2}$ and those on $\mathbf{S^2}$ is naturally induced from the inclusion map of $\mathbf{D^2}$ into $\mathbf{S^2}$. We denote by $\hat{b}$ the braid on $\mathbf{S^2}$ corresponding to $b$. Then it is natural to ask:

**Question 1.** *What happens under this correspondence?*

For example it has been determined which braids on $\mathbf{D^2}$ become trivial under the correspondence. Please refer to [1] and [8] for the theory of braids.

In this talk, we concentrate our focus on the behavior of the *Nielsen-Thurston type* under the correspondence.

### 1.1. Nielsen-Thurston type.

In the following, we give a rough explanation on the Nielsen-Thurston types of braids.

First recall that the definition for surface automorphisms. Let $\Sigma_{g,p,b}$ denote a compact orientable surface of genus $g$ with $p$ distinguished points and $b$ boundary components.

*Date*: 2004/1/31 10:00–10:30, Osaka City University, Fuculty of Science Building, room 3040.

**Definition 1** ([11]; [3], [2])**.** An orientation preserving homeomorphism of $\Sigma_{g,p,b}$ is;

(1) *periodic* if whose some power is equal to the identity,

(2) *reducible* if it leaves an essential 1-submanifold of $\Sigma_{g,p,b}$ invariant (a 1-submanifold of $\Sigma_{g,p,b}$ is called *essential* if each component is homotopically non-trivial and not boundary-parallel, and no two components are homotopic),

(3) *pseudo-Anosov* if for the map $f$, there exist a pair of transverse measured foliations $(\mathcal{F}^s, \mu^s)$, $(\mathcal{F}^u, \mu^u)$ such that $f(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda\mu^s)$ and $f(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda^{-1}\mu^u)$ for some $\lambda > 1$.

Mainly due to Nielsen and Thurston, the following trichotomy has been established.

**Fact 1** (Nielsen-Thurston classification ([11]. see also [3] or [2].))**.** *Suppose that* $2g-2+p+b > 0$ *holds. Then any orientation preserving homeomorphism of* $\Sigma_{g,p,b}$ *is isotopic to a periodic map, a reducible map, or a pseudo-Anosove map.*

Remark that this trichotomy is invariant under conjugation.

Now we give a definition of the Nielsen-Thurston types of braids.

**Definition 2.** Let $b$ be an $n$-braid either on the 2-disk $\mathbf{D^2}$ or on the 2-sphere $\mathbf{S^2}$ with $n \geq 3$.

- There exists a horizontal-level preserving homeomorphism $\Phi$ of $\mathbf{D^2} \times [0,1]$ or $\mathbf{S^2} \times [0,1]$ such that $\Phi(x,1) = (x,1)$ and $\Phi(b)$ become the trivial braid. Then $\Phi|_{\mathbf{D^2}\times\{0\}}$ or $\Phi|_{\mathbf{S^2}\times\{0\}}$ yields a homeomorphism of $\mathbf{D^2}$ or $\mathbf{S^2}$, which is determined up to isotopy. We call this homeomorphism *the homeomorphism associated to* $b$ and denote it by $f_b$.
- The braid $b$ is *periodic*, *reducible*, or *pseudo-Anosov* if $f_b$ is isotopic to a periodic map, a reducible map, or a pseudo-Anosov map, respectively.

It can be easily seen that

- if a braid $b$ on $\mathbf{D^2}$ is periodic, then the corresponding braid $\hat{b}$ on $\mathbf{S^2}$ is periodic,
- if a braid $b$ on $\mathbf{D^2}$ is reducible, then the corresponding braid $\hat{b}$ on $\mathbf{S^2}$ is reducible or $\hat{b}$ is equivalent to a conjugate of a braid which has one isolated string.

Therefore the following question arise.

**Question 2.** *For which pseudo-Anosov braid* $b$ *on* $\mathbf{D^2}$*, is* (*not*) *the corresponding braid* $\hat{b}$ *on* $\mathbf{S^2}$ *pseudo-Anosov?*

1.2. **Problem.** Concerning this question, the next observation was given by J. Los (in private communication).

**Fact 2** ([7])**.** *Let $b$ be a pseudo-Anosov braid on $\mathbf{D^2}$. If the corresponding braid $\hat{b}$ on $\mathbf{S^2}$ is NOT pseudo-Anosov, then the invariant measured foliation of $f_b$ has a 1-prong singularity on the boundary $\partial\mathbf{D^2}$.*

For example, we have the following for 3-braids on $\mathbf{D^2}$,

**Fact 3** ([9], see also [10])**.** *Let $b$ be a 3-braid on $\mathbf{D^2}$.*

(1) *The corresponding braid $\hat{b}$ of $\mathbf{S^2}$ is always periodic, in particular, is not pseudo-Anosov.*
(2) *The braid $b$ is pseudo-Anosov if and only if it is conjugate to a braid $(\sigma_1\sigma_2)^{2k}P(\sigma_1^{-1},\sigma_2)$ for some integer $k$ and a positive word $P$.*
(3) *When $b$ is pseudo-Anosov, then the invariant measured foliation of $f_b$ has a 1-prong singularity on the boundary $\partial\mathbf{D^2}$.*

In the fact above, $\sigma_1,\ldots,\sigma_{n-1}$ denote the standard Artin generators of $n$-braids on $\mathbf{D^2}$.

Thus the next problem can be considered.

**Question 3.** *Is the converse of Fact 2 true? That is, for a pseudo-Anosov braid $b$ on $\mathbf{D^2}$, if the invariant measured foliation of $f_b$ has a 1-prong singularity on the boundary $\partial\mathbf{D^2}$, then is the corresponding braid $\hat{b}$ on $\mathbf{S^2}$ not pseudo-Anosov?*

## 2. Results

Our main result is as follows: Question 3 is negatively answered.

**Theorem.** *Let $b_{n,k}$ be the $n$-braid on $\mathbf{D^2}$ given by*

$$\sigma_1\sigma_2\cdots\sigma_k(\sigma_{k+1})^{-1}\cdots(\sigma_{n-1})^{-1}$$

*where $n \geq 4$ and $1 \leq k \leq n-2$.*

(1) *The braid $b_{n,k}$ is pseudo-Anosov for all $n \geq 4$ and $1 \leq k \leq n-2$.*
(2) *The invariant measured foliation of $f_{b_{n,k}}$ has a 1-prong singularity on the boundary $\partial\mathbf{D^2}$.*
(3) *The braid $\widehat{b_{n,k}}$ on $\mathbf{S^2}$ is periodic if and only if $n$ is odd and $k = (n-1)/2$.*
(4) *The braid $\widehat{b_{n,k}}$ on $\mathbf{S^2}$ is reducible if and only if $n$ is even and $k = (n-2)/2, n/2$.*

**Corollary.** *Let $b_{n,k}$ be the $n$-braid on $\mathbf{D^2}$ as in Theorem 2. Then the corresponding braid $\widehat{b_{n,k}}$ on $\mathbf{S^2}$ is not pseudo-Anosov if and only if $k = (n-2)/2, (n-1)/2, n/2$.*

We end this note by giving some keys to our proof.

**(1) and (2)** These are achieved by actual constructions of invariant transverse measured foliations for $f_{b_{n,k}}$. Essentially this was done in [5].

**the 'if' part of (3) and (4)** One can check these part by 'hand'; by drawing and manipulating figures.

**the 'only if' part of (3)** The fact we use here is: if $\widehat{b_{n,k}}$ on $\mathbf{S^2}$ is periodic, then it is conjugate to the braid presented by

$$({\sigma_1}^{\varepsilon}{\sigma_2}^{\varepsilon}\cdots{\sigma_{n-2}}^{\varepsilon}{\sigma_{n-1}}^{\varepsilon})^m$$

where $\varepsilon = 1$ or $-1$, and $m \in \mathbb{Z}$. Then the assertion follows from [8, Chap. 11, Proposition 2.3].

**the 'only if' part of (4)** By identifying $\mathbf{S^2} \times \{0\}$ and $\mathbf{S^2} \times \{1\}$ of $\mathbf{S^2} \times [0,1]$, we obtain a knot $\widehat{K_{n,k}}$ from $\widehat{b_{n,k}}$ in $\mathbf{S^2} \times S^1$. Note that if $\widehat{b_{n,k}}$ on $\mathbf{S^2}$ is reducible, then the complement $\mathbf{S^2} \times [0,1] \setminus \widehat{K_{n,k}}$ contains an essential torus.

Let $L_{n,k}$ be the 2-component link in the 3-sphere represented as the closure of the $(n+1)$-braid

$$\sigma_1\sigma_2\cdots\sigma_{n-1}\sigma_n \cdot \sigma_1\sigma_2\cdots\sigma_k(\sigma_{k+1})^{-1}\cdots(\sigma_{n-1})^{-1}\cdot\sigma_n\sigma_{n-1}\cdots\sigma_2\sigma_1.$$

Then it is easily seen that the complement $\mathbf{S^2} \times [0,1] \setminus \widehat{K_{n,k}}$ is homeomorphic to the 3-manifold which is obtained by 0-Dehn filling on one component of $L_{n,k}$. Thus it suffices to study the toroidal Dehn surgeries on one component of $L_{n,k}$. Moreover we note that the link $L_{n,k}$ is a two-bridge link: It has the Conway form $C(2n-2k-1, 2k+1)$. Then, based on the work [4], we can show that 0-Dehn filling on one component of $L_{n,k}$ is toroidal if and only if $n$ is even and $k = (n-2)/2, n/2$. We remark that similar results were obtained in [6] independently.

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