Invariants of 3-manifolds formulated on their presentatinos given by 3-fold branched covering spaces over the 3-sphere

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1. Introduction

By a well known theorem of Hilden[3] and Montesinos [4], every closed oriented 3-manifold is a 3-fold simple branched covering of S^3 branched along a link. Piergallini [5] introduced the covering moves, which relate two such branch sets representing the same 3-manifold. Hence we can regard the branch set as a presentation of the 3-manifold. In this note we give new invariants of 3-manifolds formulated on the presentations, which are analogous to the quandle cocycle invariant introduced in [1]. Then we construct the Dijkgraaf-Witten invariant as one of the invariants.

2. Covering presentation

For a 3-manifold M a map $p : M \to S^3$ will be a *3-fold branched covering* branched along a link $L \subset S^3$, if

(1) the restriction $p : M - p^{-1}(L) \to S^3 - L$ is a usual 3-fold covering, and
(2) any point $x \in f^{-1}(L)$ has a neighbourhood homeomorphic to $\mathcal{D}\times\mathcal{I}$, where $\mathcal{D}$ is the unit disc in $\mathbb{C}$ and $\mathcal{I}$ is an interval, on which p has the form $p : \mathcal{D} \times I \to \mathcal{D} \times I, (z, t) \mapsto (z^n, t)$, for $n \in \{1, 2, 3\}$.

Such a link L is called the *branch set* of p. To any 3-fold branched covering $p : M \to S^3$, we can assign a homomorphism $\pi_1(S^3 - L) \to \mathfrak{S}_3$, where L is the branch set. A 3-fold branched covering is said to be *simple* if its assigned homomorphism maps each Wirtinger generater to a transposition.

A *diagram with transpositions* is defined to be a diagram of the branch set of a 3-fold simple branched covering. Note that this diagram is unoriented. Each arc of a diagram with transpositions is associated with a transposition (12), (23) or (31) from its assigned homomorphism. The association is not arbitrary because of the Wirtinger relations at crossings. The transpositions of an over-arc and two under-arcs at a crossing are all the same or all distinct; see Figure 1. In the following figures, the transpositions can be exchanged with each other.

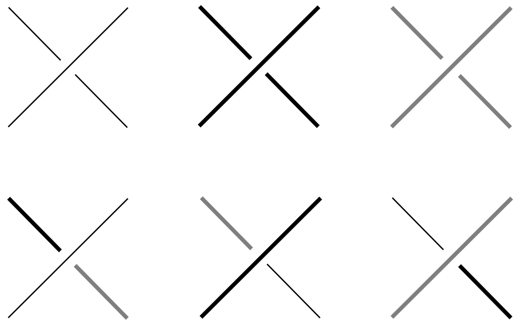


Figure 1. Crossings of diagrams with transpositions. The transpositions (12), (23) and (31) are denoted by thin lines, thick lines and gray lines, respectively.

Hilden [3] and Montesinos [4] showed that any closed oriented 3-manifold is homeomorphic to a 3-fold simple branched covering of S^3 branched along a link. Further Piergallini [5] showed that two 3-manifolds are homeomorphic iff their diagrams with transpositions are related by a finite sequence of the 3-move (Figure 2) and the covering moves (Figure 3), up to Reidemeister moves with transpositions. Hence we can regard a diagram with transpositions as a presentation of a 3-manifold. We call it the *covering presentation* of 3-manifolds. Here we have a question;

Make an invariant of 3-manifolds
formulated on their covering presentations.

For the answer of this question, we give a condition of a map which gives an invariant in Section 2, and construct the Dijkgraaf-Witten invariant by giving such a map concretely in Seciton 3.

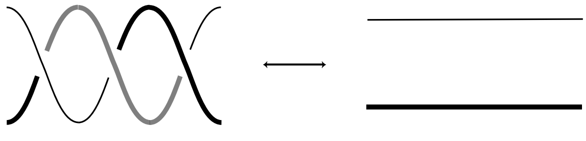


Figure 2. The 3-move

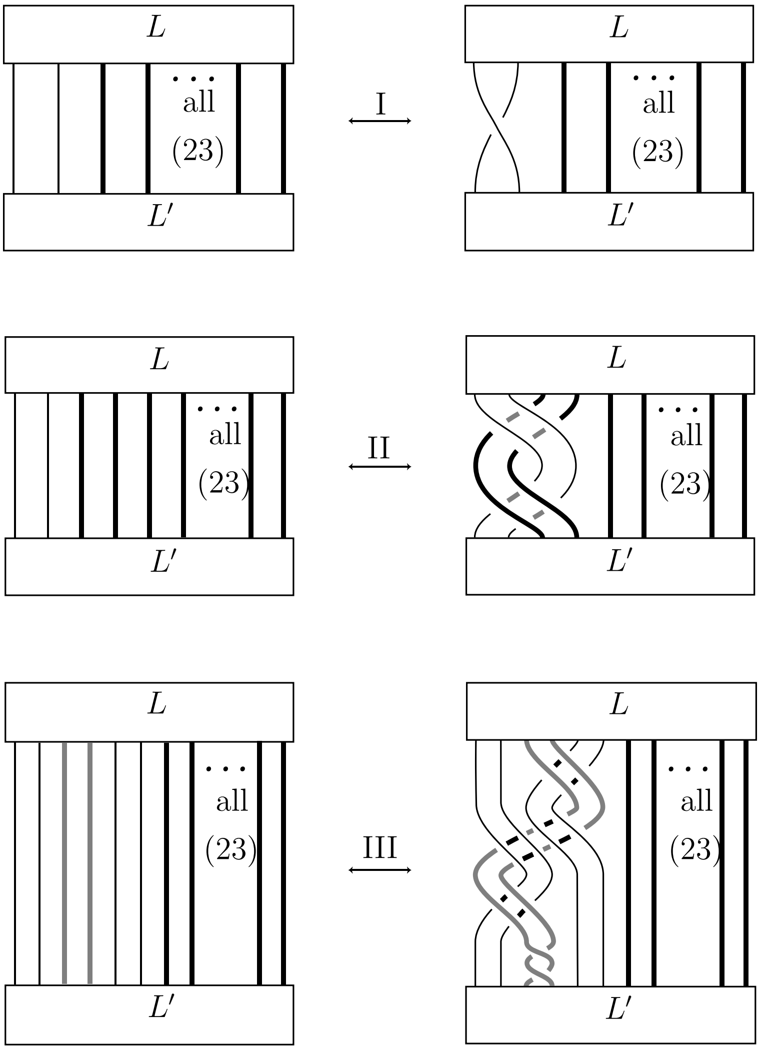


FIGURE 3. The covering moves. All the diagrams consist of a braid joining two arbitrary diagrams with transpositions L and L'.

3. INVARIATS

For a 3-manifold M, we can compute $\pi_1(M)$ from its covering presentation. Refer to [6] for more details. Using this idea, we define a coloring on a diagram with transpositions which corresponds to a representation from $\pi_1(M)$ to a finite group.

Definition 3.1. Let G be a finite group wtitten multiplicatively, D a diagram with transpositions. A *coloring* on D in G is defined to be a map $\chi : \{\text{arcs of } D\} \to G$, satisfying the following conditions,

(i) at an all the same crossing, $\chi(a) \cdot \chi(b)^{-1} \cdot \chi(c) \cdot \chi(b)^{-1} = 1$, where a and c are the under-arcs and b is the over-arc, as shown in the left hand side in Figure 4.

(ii) At an all distinct crossing, $\chi(a) \cdot \chi(b) \cdot \chi(c) = 1 \in G$, where a is the arc of transposition (12), b is of (23) and c is of (31), as shown in the right hand side in Figure 4. Note that we cannot change the order of the transpositions in the equation.

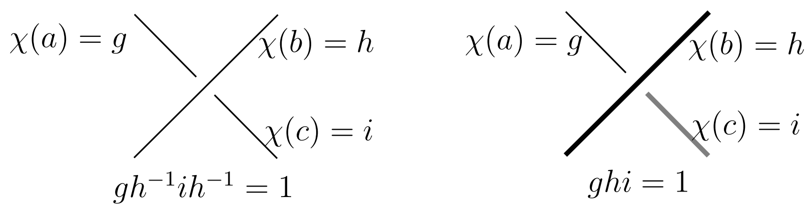


FIGURE 4. The conditions of colorings on diagrams with transpositions at an all the same crossing (left) and all distinct crossing (right)

Proposition 3.2. *The number of colorings on a diagran with transpositions in a finite group is an invariant of the 3-manifold which is given by the diagram.*

Sketch of the proof. Given a coloring on a diagram D in a finite group G, we can obtain a representation $\pi_1(M) \to G$, where M is the 3-manifold given by D. Using this representation, we can verify that the number of colorings is preserved under the Reidemeister moves with transpositions, the 3-move and the covering moves. Note that in the covering moves II and III, the colors on the arcs of L' part will be all changed in a rule. □

Proposition 3.3.

$$\sharp\{\textit{colors on } D \textit{ in } G\} = |G|^2 \cdot \sharp\{\textit{representations } \pi_1(M) \to G\}.$$

By Proposition3.2, we construct state sum invariants of 3-manifolds formulated on their covering presentations in the following way.

Let A be a set. If a map

$$X : \{\text{crossings of a colored diagram with transpositions}\} \to A$$

is invariant under the Reidemeister moves, 3-move and covering moves, then the expression

$$\sum_{\chi} \prod_{\tau} X(\tau) \in \mathbb{Z}[A],$$

where the product is taken over all crossings and the sum is taken over all possible colorings in G, is an invariant of 3-manifolds.

4. Dijkgraaf-Witten invariant

We first review the Dijkgraaf-Witten invariant of [2, 7]. Let M be a closed oriented 3-manifold with a triangulation T of N vertices. We give an ordering to the set of the vertices. A *coloring* on T in a finite group G is a map $\omega : \{\text{oriented edges of } T\} \to G$, satisfying the condition depicted in Figure 5, and $\omega(-E) = \omega(E)^{-1}$ for any edge E, where $-E$ is the edge with the opposite orientation.

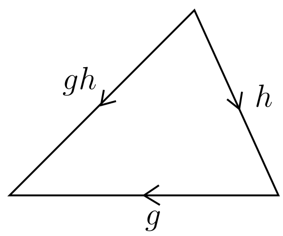


Figure 5. The condition of colorings on triangulations.

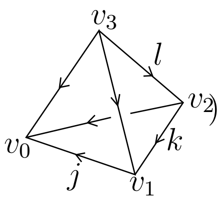
A map $\theta : G \times G \times G \to A$, where A is an abelian group written multiplicatively, is said to be a *3-cocycle* if it satisfies

$$\begin{cases} \theta(1,x,y) = \theta(x,1,y) = \theta(x,y,1) = \theta(x,x^{-1},y) = \theta(x,y,y^{-1}) = 1, \\ \theta(y,z,w) \cdot \theta(xy,z,w) \cdot \theta(x,yz,w) \cdot \theta(x,y,zw) \cdot \theta(x,y,z) = 1, \end{cases}$$

for any $x, y, z, w \in G$. Then the *Dijkgraaf-Witten invariant* is defined by

$$Z_\theta(M) = \frac{1}{|G|^N} \cdot \sum_{\omega} \prod_{\sigma;\ \text{tetrahedron}} W(\sigma) \in \mathbb{Z}[A],$$

where the product is taken over all tetrahedron, the sum is taken over all possible colorings, and

$W($ 

$) = \theta(j, k, l),$

with $v_0 < v_1 < v_2 < v_3$.

Now let us construct the Dijkgraaf-witten invariant on covering presentations. First we introduce the coloring on the regions of a diagram uniquely given by a coloring on the diagram.

Definition 4.1. Let χ be a coloring on a diagram D in a finite group G. The *coloring on the regions* given by χ is an assignment of an triplet in $G \times G \times G$ to each region of $\mathbb{R}^2 \setminus D$ with the following rules:

(a) The unbounded region is assigned $(1, 1, 1)$.

(b) Two regions separated by an arc are assigned as depicted in Figure 6.

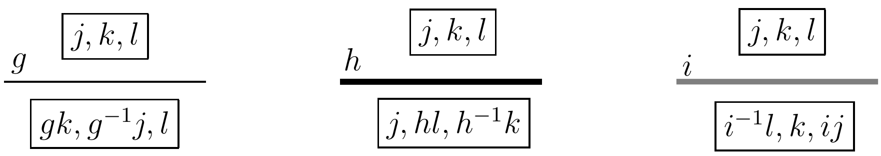
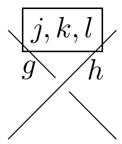


FIGURE 6. The condition of coloring on regions

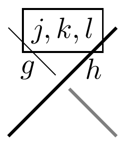
We can verify that the definition is compatible around any crossing.

Then we define the weight of crossings of a colored diagram with the coloring on regions associated with a 3-cocycle θ in the following equation;

$X($ 

$) = \theta(g, g^{-1}h, h^{-1}j) \cdot \theta(h, h^{-1}g, k)$

$$\cdot\, \theta(hg^{-1}h, h^{-1}g, g^{-1}j) \cdot \theta(h, g^{-1}h, h^{-1}gk),$$

$X_\theta($ 

$) = \theta(g^{-1}, gh, h^{-1}k) \cdot \theta(g^{-1}, j, j^{-1}gh).$

Theorem 4.2. *This X_θ satisfies the condition in the box in Section 3. Hence $\sum_\chi \prod_\tau X_\theta(\tau)$ defines an invariant of 3-manifolds.*

Proposition 4.3. *Let D be a diagram with transpositions, M the 3-manifold given by D. Then*

$$\sum_{\chi} \prod_{\tau} X_\theta(\tau) = \frac{1}{|G^3|} \cdot Z_\theta(M).$$

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