Surface braid monodromies on a punctured disk

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Abstract

We study surface braid monodromies on a punctured disk by using a monodromy system. It is shown that any monodromy system of a surface braid of degree 3 is ribbon. Here we show that there is a non-ribbon monodromy system.

1 Monodromies on a punctured disk

Let D^2 be a 2-disk and Σ be a set of n interior points in D^2. Fix a base point $y \in \partial D^2$. For a group G, a *G-monodromy* ρ is a homomorphism

$$\rho : \pi_1(D^2 \backslash \Sigma, y) \to G.$$

Two G-monodromies ρ and ρ' are *equivalent*, denoted by $\rho \sim \rho'$, if there exist a homeomorphism $h : (D^2, \Sigma, y) \to (D^2, \Sigma, y)$ and an inner automorphism $\alpha : G \to G$ such that

$$\rho' = \alpha \circ \rho \circ h_*.$$

We often want to classify G-monodromies with some additional conditions under the equivalence $\sim$. For example, Lefschetz fibrations on a sphere with G: the mapping class group on a closed surface. ([2]) or algebraic curves in a projective plane with G: the m-th braid group. ([3]).

To study monodromies easily or systematically, we use a *Hurwitz generating system* $H = (\eta_1, \eta_2, \dots, \eta_n)$ of $\pi_1(D^2 \backslash \Sigma, y)$ which satisfies the following conditions:

- each η_j surrounds one puncture in a positive direction (see the following figure); and
- $\eta_1 \cdot \eta_2 \cdot \dots \cdot \eta_n = [\partial D^2]$.

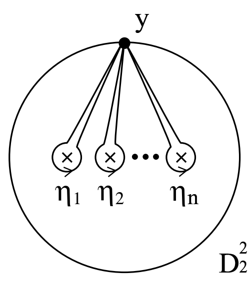

Figure 1:

A *G-monodromy system* is an n-tuple

$$\mathrm{MS}(\rho, H) := (\rho(\eta_1), \rho(\eta_2), \ldots, \rho(\eta_n)) \in G \times \cdots \times G.$$

A *Hurwitz equivalence* is a equivalence relation on $G \times \cdots \times G$ corresponding to the equivalence of monodromies, which is generated by the following relations:

$$\begin{array}{rcl} (\xi_1, \ldots, \xi_j, \xi_{j+1}, \ldots, \xi_n) & \sim & (\xi_1, \ldots, \xi_{j+1}, \xi_{j+1}^{-1}\xi_j\xi_{j+1}, \ldots, \xi_n), \\ (\xi_1, \ldots, \xi_n) & \sim & (\beta^{-1}\xi_1\beta, \ldots, \beta^{-1}\xi_n\beta), \end{array}$$

where $j = 1, 2, \ldots, n-1$ and $\beta \in G$. If monodromy systems MS and MS$'$ are Hurwitz equivalent, we denote it by $\mathrm{MS} \overset{H}{\sim} \mathrm{MS}'$.

Theorem 1 *Let ρ and ρ' be G-monodromies and H and H' are Hurwitz generating systems. Then*

- $\rho \sim \rho'$ *if and only if* $\mathrm{MS}(\rho, H) \overset{H}{\sim} \mathrm{MS}(\rho', H)$,
- $(\xi_1, \ldots, \xi_n) = \mathrm{MS}(\rho, H')$ *if and only if* $(\xi_1, \ldots, \xi_n) \overset{H}{\sim} \mathrm{MS}(\rho, H)$.

2 The surface braid monodromy

Let $D_1^2 \times D_2^2$ be a 4-disk and X_m be a set of m-interior points in D_1^2. A *simple surface braid* of degree m is an oriented compact surface S embedded properly and locally flatly in $D_1^2 \times D_2^2$ which satisfies the following conditions:

- the restriction map $pr_2|_S : S \to D_2^2$ is an m-fold branched covering map,
- $\partial S = X_m \times \partial D_2^2$, and
- $\sharp(S \cap pr_2^{-1}(y)) \geq m-1$ for any $y \in D_2^2$.

We denote a set of branch points in D_2^2 by $\Sigma(S)$. By the condition about ∂S, the number n of elements of $\Sigma(S)$ must be even. For each $y \in D_2^2 \backslash \Sigma(S)$,

$$pr_1(S \cap pr_2^{-1}(y)) \subset \mathrm{Int}(\mathrm{D}_1^2)$$

is a set of distinct m-points in the interior of D_1^2. Thus we get a homomorphism

$$\rho_S : \pi_1(D_2^2 \backslash \Sigma(S), y_0) \to B_m,$$

called a *surface braid monodromy* of S, where B_m is the m-th braid group given as the following way : for each closed curve γ in $D_2^2 \backslash \Sigma(S)$, we define the closed curve $\widetilde{\gamma}$ in the configuration space of unordered m-interior points of D_1^2

$$\widetilde{\gamma}(t) := pr_1(S \cap pr_2^{-1}(\gamma(t))).$$

The fundamental group of this configuration space is isomorphic to B_m. Now we consider

monodromy system $\mathrm{MS}(\rho_S, H) = (\rho_S(\eta_1), \rho_S(\eta_2), \ldots, \rho_S(\eta_n)) \in B_m \times \cdots \times B_m$. We say that $\mathrm{MS}(\rho_S, H)$ is *ribbon* if it is equivalent to a system $(\xi_1, \ldots, \xi_n)$ such that $\xi_{2j-1}\xi_{2j} = id_{B_n}$ for each $j = 1, 2, \ldots, \frac{n}{2}$. We say that a surface braid is *ribbon* if its monodromy system is ribbon. Here note that if a surface link has a closed surface braid presentation whose braid is ribbon, then the link is ribbon. However it is not known whether the other way holds or not. We have the following on the ribbonness of a monodromy system.

Theorem 2 ([*1*]) *Any monodromy system of a surface braid of degree* 3 *is ribbon.*

Now let φ be a map from B_m to $(\mathbb{Z}_2)^m \rtimes S_m$ given as follows and let π be the projection map from $(\mathbb{Z}_2)^m \rtimes S_m$ to S_m , where S_m is the m-th symmetric group.

$$\varphi(\sigma_i) = \begin{pmatrix} I_{i-1} & & & \\ & 0 & 1 & \\ & t & 0 & \\ & & & I_{m-i-1} \end{pmatrix} = \begin{pmatrix} I_{i-1} & & & \\ & 1 & 0 & \\ & 0 & t & \\ & & & I_{m-i-1} \end{pmatrix} \cdot \begin{pmatrix} I_{i-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_{m-i-1} \end{pmatrix}$$

Observing surface braid monodromies by using homomorphism $s = \pi \cdot \varphi$, we obtain the following theorem. As a corollary of the theorem, we can show that there is a non-ribbon monodromy system.

Theorem 3 *Let* $\mathrm{MS}(\rho_S, H) = (b_1, \cdots, b_n)$ *and* $\mathrm{MS}(\rho_{S'}, H') = (b'_1, \cdots, b'_n)$ *be ribbon monodromy systems. If* $(s(b_1), \cdots, s(b_n))$ *is Hurwitz equivalent to* $(s(b'_1), \cdots, s(b'_n))$*, then* $(\varphi(b_1), \cdots, \varphi(b_n))$ *is Hurwitz equivalent to* $(\varphi(b'_1), \cdots, \varphi(b'_n))$*.*

Corollary 4 *If* $p \equiv 2$ *and* $q \equiv 2 \pmod 4$*, then a monodromy system* $(b_1, b_2, \ldots, b_8)$ *is non-ribbon.*

$$\begin{array}{ll} b_1 = \overline{2}32, & b_5 = \overline{1}^p\overline{4}^q\overline{2}324^q1^p \\ b_2 = \overline{3}\,\overline{2}3, & b_6 = \overline{1}^p\overline{4}^q\overline{3}\,\overline{2}34^q1^p \\ b_3 = \overline{1}^p\overline{3}231^p, & b_7 = \overline{4}^q\overline{3}234^q \\ b_4 = \overline{1}^p\overline{2}\,\overline{3}21^p, & b_8 = \overline{4}^q\overline{2}\,\overline{3}24^q \end{array}$$

References

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