LECTURE NOTES FOR WORKSHOP OF FLEDGLINGS ON LOW-DIMENSIONAL TOPOLOGY

JANUARY 30 – FEBRARY 2, 2004
AT
OSAKA CITY UNIVERSITY

PREFACE

This is a collection of lecture notes and resumes for the mini-workshop 'Workshop of Fledglings[1] on Low-dimensional Topology' held at Osaka City University during January 30 – Febrary 2, 2004, as a part of the 21st century COE program "Constitution of wide-angle mathematical basis focused on knots" (Akio Kawauchi, the project reader).

This workshop was organized principally for Post Doctors and Doctor course students studying Low-Dimensional Topology and its related topics. It was the main aim of the workshop to provide them opportunities; to introduce and discuss their recent results each other, and; to make and deepen their friendship.

The workshop consisted of 4 half-days lectures and 10 short talks. The participants were 45 or more, and there, a lot of animated discussions have been done.

The organizers primely would like to thank all speakers for providing interesting talks and for preparing well-written resumes. They also thank to Tomomi Murai, Yuki Tadokoro, Tatsuya Tsukamoto, and Reiko Shinjo for their support to produce this collection of notes.

Organizers:
Kazuhiro Ichihara (Nara Women's University, JSPS Research Fellow)
Takuji Nakamura (OCAMI, Osaka City University)
Ryo Nikkuni (Waseda University, JSPS Research Fellow)

[1]**fledgling** 1. A young bird that has recently acquired its flight feathers. 2. A young or inexperienced person.

Program

January 30

13:00—16:15 Teruhisa Kadokami (OCAMI, Osaka City University)
How to use the Reidemeister torsion

16:30—17:30 Nafaa Chbili (Tokyo Institute of Technology)
Invariants of freely periodic knots
(under the joint auspices of OCAMI, room 3153)

17:45— Teruhisa Kadokami (OCAMI, Osaka City University)
How to use the Reidemeister torsion

January 31

10:00—10:30 Kazuhiro Ichihara (Nara Women's University, JSPS Research Fellow)
Pseudo-Anosov braids on the 2-sphere

10:45—11:15 Naoko Tamura (Tokyo Metropolitan University)
On the A-polynomial of a knot

11:30—12:00 Ryosuke Yamamoto (Osaka University)
Legendrian curves on fiber surfaces

13:30—18:00 Tetsuhiro Moriyama (University of Tokyo, JSPS Research Fellow)
On the configuration space of points and the Casson invariant

18:30— Party

February 1

10:00—10:30 Yukihiro Tsutsumi (Keio University, JSPS Research Fellow)
A surgery description of homology solid tori
and its applications to the Casson invariant

10:45—11:15 Norihisa Teshigawara (Tokyo Institute of Technology)
On the Khovanov invariant for links

11:30—12:00 Eri Hatakenaka (Tokyo Institute of Technology)
Invariants of 3-manifolds formulated on their presentations
given by 3-fold branched covering spaces over the 3-sphere

13:30—18:00 Takahito Kuriya (Kyushu University)
A proof of the LMO conjecture

February 2

10:00—10:30 Kokoro Tanaka (University of Tokyo)
Braid indices of surface-knots and colorings by quandles

10:45—11:15 Isao Hasegawa (University of Tokyo)
Monodromies on a punctured disk

11:30—12:00 Takashi Makino (Kobe University)
Delta-unknotting numbers and the Conway polynomials of knots

13:30—17:30 Ryo Nikkuni (Waseda University, JSPS Research Fellow)
Delta link-homotopy on spatial graphs

Contents

How to use the Reidemeister torsion

Teruhisa Kadokami (OCAMI, Osaka City University)

Reidemeister torsion の利用法

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Abstract. Firstly, we give the definition of the Reidemeister torsion, and explain basic properties, following V. G. Turaev. Secondly, we consider the Reidemeister torsion of a homology lens space, which is the result of p/q-surgery along a knot K in a homology 3-sphere Σ. We denote the homology lens space by $\Sigma(K;p/q)$. Main Theorem 1 is the case that K is a torus knot in S^3. Main Theorem 2 is the case that the Alexander polynomial of K, $\Delta_K(t)$, is degree 2. We judge when the homology lens spaces are homeomorphic to lens spaces by using the Reidemeister torsion.

Abstract

1935 年、K. Reidemeister [20] は閉 3 次元多様体に対して torsion invariant を定義して、3 次元 lens space を完全に分類した．この invariant を我々は *Reidemeister torsion* と呼ぶ．W. Franz [6] は一般次元の lens space を分類した．やはり torsion invariant を用いたのだが、その際代数的数論的な定理が決め手になっている．我々の第 1 の結果は、この Franz の定理を応用している．この強力な定理は、意外と利用されていないと思われる．Franz の定理の証明には L 関数の理論が用いられていて、近年結び目理論と数論の関係が深くなってきている流れの中で、その応用や一般化などは掘り下げる価値のある問題だろう．

1962 年、J. W. Milnor [14] は Reidemeister torsion と Alexander polynomial の密接な関係を指摘した．V. G. Turaev [24], [25], [26] は 1976 年からの一連の論文で、有限 CW 複体の Reidemeister torsion を計算し、特にコンパクト 3 次元多様体に対して詳しい計算公式を与えた．

前半は、V. G. Turaev に従って Reidemeister torsion を定義し、基本的な計算公式を与える．本質は、homology theory における Mayer-Vietoris の定理に対応する切除性質が Reidemeister torsion に対しても成り立つことである．ただし、Dehn surgery とは相性がいいが、Heegaard 分解とは相性があまりよくないことは注意しておく．なぜなら、Reidemeister torsion が 0 でないための必要条件は Euler number が 0 であることだからである．

後半は、homology lens space に的を絞って、Reidemeister torsion の値そのものを見ていき、lens space の Reidemeister torsion の値と一致するかどうかを見ていく．一致するときを *lens space type* と定義する．“値そのものを見ていく” の意味だが、

値の表示は割と簡単にできるのだが、2 つ表示された値を持ってきたとき、それらが同じか違うかの判定が意外と難しい．Homology lens space の Reidemeister torsion の値は円分体に値を取るので、当然円分体（に限らない代数体）に関する代数的数論の知識が必要になってくる．しかし筆者のレベルからしてまだ大したことはできていないが、今後（この勉強会を通して）他の人々の手によって高いレベルの応用がなされることを期待している．

今回の我々の結果は、地道な幾何的な手法によって得られた精密な結果を Reidemeister torsion で見直していくものである．これには大きく 3 つの意図がある．1 つ目は、Reidemeister torsion の invariant としての切れ味を見るものである．どれぐらい精密なのか？それ程でもないのか？2 つ目は、トポロジーにおける幾何的手法と代数的手法の関係として、代数的手法は偵察部隊の役割を持つべきだが（特に、精密さをある程度犠牲にした invariant は）、近頃はやや幾何的手法が先行している向きがあるので、或いは、精密な invariant は偵察部隊というより主力部隊の役割を担おうとする向きがあるので、Reidemeister torsion に偵察部隊としての役割を担ってもらいたい意図がある．3 つ目は、より精密な invariant の技術開発をも刺激するであろうことである．

我々の第 1 の結果は、L. Moser [16] (1971) による torus knot に沿った rational surgery の結果の多様体の完全な分類を Reidemeister torsion で翻訳する．Lens space に対しては精密さを保っていた Reidemeister torsion は、この homology lens space に対しては必ずしも精密ではないことがわかる．しかし偵察部隊としては（筆者の偏見込みで）合格点を与えられる．

第 2 の結果は、合田-寺垣内 [7] (2000) による結果の一部である、genus 1 knot K に沿った surgery の結果が lens space になるならば、K は trefoil である、という結果の翻訳をする．我々は、次数 2 の Alexander polynomial を持つ knot K に沿う surgery の結果が lens space type のとき、K の Alexander polynomial が $t^2 - t + 1$ であることを示した．

第 1 の結果の証明は Franz の定理を使い、第 2 の結果の証明は代数的数のノルムを利用した．これらの手法はそれぞれ拡張の可能性を持っている．最後にその拡張と応用を行った．（今回は書かない．次ページ参照.）まだまだやるべきことは残っている．また、Ozsváth-Szabó の結果 [17], [11] (2003) との関係も興味のある所である．

3 次元 lens space の分類定理の別証明は [1], [19]、simple homotopy や Whitehead torsion については [3], [15], [30]、twisted Alexander polynomial については [28] を参照されたい．

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（お詫び）実際の講演は、1 章は割と丁寧に、3 章はアウトラインのみの説明でした. 予定は上記にあるものでしたが、4 章は現在進行中あるいは進行予定の面があるために、割愛させていただきます. 現在進行中の一部は [10] にありますので、そちらを参照してください.

0. Statement of Main Theorems

細かい用語は後述する．動機となった定理と我々の 2 つの結果を述べる．

Theorem (Moser [16]; Gordon [8]; Shimozawa [23]) *Let $K_{r,s}$ be the (r,s)-torus knot in S^3, and $M = S^3(K_{r,s}; p/q)$ the result of p/q-surgery along $K_{r,s}$ where $|p|, |r|, |s| \geq 2$ and $q \neq 0$. Then there are three cases :*

(1) *If $|p - qrs| \neq 0$, then M is a Seifert fibered space with three singular fibers of multiplicities $|r|, |s|$ and $|p - qrs|$. In particular,*

(2) *if $|p - qrs| = 1$, then M is the lens space $L(p, qr^2)$* (Figure 1).

(3) *If $|p - qrs| = 0$ $(p/q = rs)$, then M is the connected sum of two lens spaces, $L(r,s)\sharp L(s,r)$.*

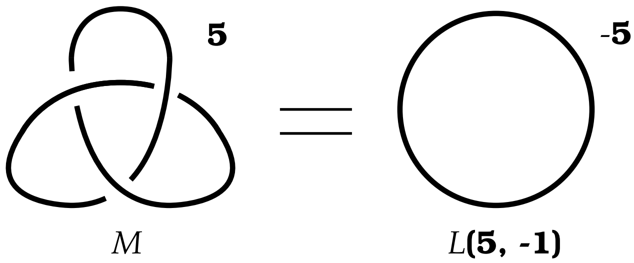

FIGURE 1. the result of surgery is a lens space

Notation

$$\Delta_{r,s}(t) := \frac{(t^{rs}-1)(t-1)}{(t^r-1)(t^s-1)} \qquad ((r,s)=1),$$

ここで (r,s) は r と s の最大公約数を表す．$(r,s) = 1$ は r と s が互いに素ということである．

Main Theorem 1 *Let $K_{r,s}$ $((r,s) = 1)$ be a knot in a homology 3-sphere Σ with its Alexander polynomial $\Delta_{r,s}(t)$, and $M = \Sigma(K_{r,s}; p/q)$ the result of p/q-surgery along $K_{r,s}$ where $|p|, |r|, |s| \geq 2$ and $q \neq 0$. Then M is of lens space type if and only if the following (1) and (2) hold.*

(1) *$(p,r) = 1$, and $(p,s) = 1$, and*

(2) *$r \equiv \pm 1 \pmod p$ or $s \equiv \pm 1 \pmod p$ or $qrs \equiv \pm 1 \pmod p$.*

Theorem (Goda-Teragaito [7]) *Let K be a genus 1 knot in S^3. If a rational surgery along K yields a lens space, then K is the trefoil.*

Notation $\Delta_n(t) := n(t-1)^2 + t = nt^2 - (2n-1)t + n \quad (n \neq 0).$

Main Theorem 2 *Let K be a knot in a homology 3-sphere Σ with its Alexander polynomial $\Delta_K(t) = \Delta_n(t)$, and $M = \Sigma(K; p/q)$ the result of p/q-surgery along K where $|p| \geq 2$ and $q \neq 0$. Let d (≥ 2) be a divisor of p, ξ_d a primitive d-th root of unity, $\psi_d : \mathbf{Z}[t, t^{-1}]/(t^p - 1) \to \mathbf{Q}(\xi_d)$ a homomorphism such that $\psi_d(t) = \xi_d$, and $\tau^{\psi_d}(M)$ the Reidemeister torsion associated to ψ_d. Then the following (1) and (2) hold.*

(1) *If $n \leq -1$, then $\tau^{\psi_p}(M)$ is not of lens space type.*

(2) *If $|n| \geq 2$ and d is a prime number, then $\tau^{\psi_d}(M)$ is not of lens space type.*

これにより以下が導かれる.

Corollary *In the same assumption as Main Theorem 2, if M is of lens space type, then*

$$\Delta_K(t) = t^2 - t + 1 \quad (n = 1).$$

これは合田-寺垣内 [7] の結果の代数的翻訳になっていて、以下の Ozsváth-Szabó [17] の結果の一部の拡張もしている.

Theorem (Ozsváth-Szabó [17]) *Let K be a knot in S^3, and $M = S^3(K; p)$ the result of p-surgery along K where p is an integer. If M is a lens space, then the Alexander polynomial of K is the following form*

$$\Delta_K(t) = (-1)^m + \sum_{j=1}^{m} (-1)^{m-j}(t^{s_j} + t^{-s_j}),$$

where $0 < s_1 < s_2 < \cdots < s_m$.

Terminology （基本となる用語集）

- Reidemeister torsion $(\tau(\mathbf{C}_*), \tau^\varphi(\mathbf{C}_*), \tau(X), \tau^\varphi(X))$: ある環 R 上 finitely generated free chain complex $\mathbf{C}_*$ から決まる invariant のことをいう．記号で $\tau(\mathbf{C}_*)$ と表す．値は R の商体の元である．実際に値が計算できるように様々な制限を付ける．環 R は integral domain である方が扱い易い．環 R から計算しにくいときは、環準同型 $\varphi : R \to R'$ に付随した invariant $\tau^\varphi(\mathbf{C}_*)$ を計算するのがよい．本講演の肝となるテクニックである．ここで、R' は invariant が計算し易い環である．この値も Reidemeister torsion という．

空間の Reidemeister torsion は、finite CW-complex X に対して定義される．X の cell decomposition からの自然な chain complex が存在する．それから Reidemeister torsion を計算するのもいいが、そうではなくて、X の適当な covering space $\tilde{X}$ の cell decomposition からの自然な chain complex の Reidemeister torsion を計算する．本講演では $\tilde{X}$ としては、maximal abelian covering を取ってくる．$H = H_1(X; \mathbf{Z})$ が covering transformation group になり、$\tilde{X}$ からの chain complex $\mathbf{C}_*(\tilde{X})$ にも H の元が作用し、$\mathbf{Z}[H]$-chain complex と見なせる．この Reidemeister torsion を X の Reidemeister torsion として、$\tau(X)$ と表す．実際は環準同型 $\varphi : \mathbf{Z}[H] \to R$ に付随した $\tau^\varphi(X)$ にしないと意味のある値が出てきにくい．そして、この invariant には自由度がある．$\pm\varphi(H)$ の元の倍数は同じ値と見なす．Alexander polynomial $\Delta_K(t)$ には $\pm t^n$ 倍の自由度があるのと同じ理由である．本講演では、さらに "隠れた自由度" があることを指摘し、追求していく．

- Homology lens space $(M = \Sigma(K; p/q))$: 向き付け可能閉 3 次元多様体 M が *homology lens space* であるとは、$H_1(M; \mathbf{Z})$ が有限巡回群 $\mathbf{Z}/p\mathbf{Z}$ のときをいう．本講演では $p \geq 2,\ p \neq \infty$ を仮定する．

任意の homology lens space M は、適当な homology 3-sphere Σ 内の適当な knot K に沿った適当な p/q-surgery の結果として表すことができる．ここで、$|p| \geq 2, q \neq 0$. 記号で $M = \Sigma(K; p/q)$ と表す．

- Lens space type : Lens space $L(p, q)$ の 1 次元ホモロジー群の生成元を t とする．つまり、$H = H_1(L(p, q); \mathbf{Z}) = \langle t \rangle = \mathbf{Z}/p\mathbf{Z}$. ξ_d を 1 の原始 d 乗根とする．ここで、$d\ (\geq 2)$ は p の約数．準同型 $\psi_d : \mathbf{Z}[H] \to \mathbf{Q}(\xi_d)$ を $\psi_d(t) = \xi_d$ から決まるものとする．このとき、$\tau^{\psi_d}(L(p, q)) = (\xi_d - 1)^{-1}(\xi_d^{\bar{q}} - 1)^{-1}$ である．ここで、$q\bar{q} \equiv 1 \pmod{p}$.

ζ を 1 の原始 n 乗根とするとき、第 n 次円分体 $\mathbf{Q}(\zeta)$ の元 a が *lens space type* であるとは、$a = \pm\zeta^m(\zeta^i - 1)^{-1}(\zeta^j - 1)^{-1}$ $((i, n) = 1,\ (j, n) = 1)$ と表されるときをいう．a はある lens space の Reidemeister torsion の値になる．

Homology lens space $M = \Sigma(K; p/q)$ の 1 次元ホモロジー群の生成元を t とする．つまり、$H = H_1(M; \mathbf{Z}) = \langle t \rangle = \mathbf{Z}/p\mathbf{Z}$. ξ_d を 1 の原始 d 乗根とする．ここで、$d\ (\geq 2)$

は p の約数. 準同型 $\psi_d : \mathbf{Z}[H] \to \mathbf{Q}(\xi_d)$ を $\psi_d(t) = \xi_d$ から決まるものとする. このとき、いかなる d に対しても $\tau^{\psi_d}(M)$ が lens space type であるとき、M そのものを *lens space type* という. もはや Reidemeister torsion では lens space との差を判定することができない多様体ということである.

• 代数的数のノルム $(N_{K/\mathbf{Q}}(\alpha))$: $K/\mathbf{Q}$ を $\mathbf{Q}$ 上の有限次 Galois 拡大とする. K の元 α の $\mathbf{Q}$ 上のノルム $N_{K/\mathbf{Q}}(\alpha)$ を以下のように定義する.

$$N_{K/\mathbf{Q}}(\alpha) = \prod_{\sigma \in \mathrm{Gal}\,(K/\mathbf{Q})} \sigma(\alpha).$$

ここで、$\mathrm{Gal}(K/\mathbf{Q})$ は Galois 拡大 $K/\mathbf{Q}$ の Galois 群とする. α の最小多項式を monic にしたときの定数項のベキ乗の ± 1 倍になっている.

1. Definition of Reidemeister torsion

Reidemeister torsion の定義は、代数的な準備をしてから finite CW-complex X に対してなされる. 以下、Turaev [25] に従って定義する.

1.1 Reidemeister torsion of chain complex

以下に出てくる環は全て 1 を持つ可換環 で零環でないもの（$1 \neq 0$）とする.

Definition 1.1. 環 R 上の chain complex $\mathbf{C}_*$ が *finitely generated* とは、C_i $(i = 0, 1, \cdots, m)$ が finitely generated R-module で、

$$\mathbf{C}_*: \ 0 \to C_m \xrightarrow{\partial_{m-1}} C_{m-1} \xrightarrow{\partial_{m-2}} \cdots \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_0} C_0 \to 0$$

であるときをいう. $\mathbf{C}_*$ が *free* とは、C_i $(i = 0, 1, \cdots, m)$ が全て free R-module のときをいい、$\mathbf{C}_*$ が *acyclic* とは、$\mathbf{C}_*$ が exact sequence のときをいう.

$$\mathbf{C}_* \text{ が acyclic} \Longleftrightarrow H_*(\mathbf{C}_*) = \bigoplus_{i=0}^{m} H_i(\mathbf{C}_*) = 0.$$

以下 $\mathbf{C}_*$ は、はじめと最後の 0 を省略して、

$$\mathbf{C}_*: \ C_m \xrightarrow{\partial_{m-1}} C_{m-1} \xrightarrow{\partial_{m-2}} \cdots \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_0} C_0$$

と表すこともある.

主に環 R が、以下の性質 $(*)$ を満たす場合を考えていく.

$(*)$ R 上の finitely generated free module M の basis の濃度は一定.

まず M の basis の濃度が有限なのは容易にわかる.（背理法で示すことができる.）$(*)$ を満たす十分条件として、次のようなものがある.

Proposition 1.2. [3] 環 R から integral domain K への homomorphism $f : R \to K$ で $f(1_R) = 1_K$ となるものがあるとき、R は $(*)$ を満たす.

Proof R 上の finitely generated free module を M とする. M の 2 つの basis $\{x_i\}_{i=1,\cdots,m}$, $\{y_j\}_{j=1,\cdots,n}$ を用意する. このとき、$m = n$ を示せばよい.

$$x_i = \sum_{j=1}^{n} a_{ij} y_j \ (i = 1, \cdots, m), \ y_j = \sum_{i=1}^{m} b_{ji} x_i \ (j = 1, \cdots, n) \text{ とすると、}$$

$$A = \begin{pmatrix} a_{ij} \end{pmatrix}_{i=1,\cdots,m \ ; \ j=1,\cdots,n}, \ B = \begin{pmatrix} b_{ji} \end{pmatrix}_{j=1,\cdots,n \ ; \ i=1,\cdots,m}$$

は変換行列である. これは、$AB = I_m$, $BA = I_n$ (I_m, I_n は m 次、n 次の単位行列) を満たす.

$f(I_m) = I_m$, $f(I_n) = I_n$ より、$f(AB) = f(A)f(B) = I_m$, $f(BA) = f(B)f(A) = I_n$.

K の商体 $Q(K)$ でも同じ関係なので、$m = n$. □

Corollary 1.3. [3] group G で生成する group ring $\mathbf{Z}[G]$ は $(*)$ を満たす.

Proof G の元を 1 に移す準同型 $f : \mathbf{Z}[G] \to \mathbf{Z}$ が Proposition 1.2 の条件を満たす. □

以下に出てくる環は <u>全て $(*)$ を満たす</u> とする.

Notation (1) 環 R 上の free module M の basis の濃度を、M の *dimension* または *rank* といい、$\dim_R M$ ($\dim M$) または $\mathrm{rank}_R M$ ($\mathrm{rank}\, M$) と表す. （Turaev は、$\mathrm{rk}_R M$ ($\mathrm{rk}\, M$) と表している.）

(2) M の 2 つの basis を $\mathbf{b} = (b_1, b_2, \cdots, b_r), \mathbf{c} = (c_1, c_2, \cdots, c_r)$ とする. ここで、$r = \mathrm{rank}\, M$.

$b_i = \displaystyle\sum_{j=1}^{r} a_{ij} \cdot c_j \ (i = 1, \cdots, r)$ となる $a_{ij} \in R \ (i, j = 1, \cdots, r)$ が存在する.

$A = \begin{pmatrix} a_{ij} \end{pmatrix}_{i,j=1,\cdots,r}$ とするとき、$\left[\mathbf{b}/\mathbf{c}\right] := \det(A)$ とする.

$r = 0$ のときは $\left[\emptyset/\emptyset\right] = 1$ とする.

このとき以下がわかる. $\mathbf{b}, \mathbf{c}, \mathbf{d}$ を free module M の basis とする.

① $\left[\mathbf{b}/\mathbf{c}\right]$ は R の正則元である.

② $\left[\mathbf{b}/\mathbf{c}\right] \cdot \left[\mathbf{c}/\mathbf{d}\right] = \left[\mathbf{b}/\mathbf{d}\right]$.

③ $\left[\mathbf{b}/\mathbf{b}\right] = 1, \ \left[\mathbf{c}/\mathbf{b}\right] = \left[\mathbf{b}/\mathbf{c}\right]^{-1}$.

④ R の正則元 u と $\mathbf{c}$ が与えられたとき、$\left[\mathbf{b}/\mathbf{c}\right] = u$ となる $\mathbf{b}$ が存在する.

①, ②, ③ により、$\left[\mathbf{b}/\mathbf{c}\right] = 1$ のとき、$\mathbf{b}$ と $\mathbf{c}$ は *equivalent* と定義すると、同値関係になる.

(3) $\mathbf{b}' = (b_1, b_2, \cdots, b_{r'}), \mathbf{b}'' = (b_{r'+1}, \cdots, b_r)$ のとき、$\mathbf{b} = \mathbf{b}'\mathbf{b}'' = (b_1, b_2, \cdots, b_r)$ と表す.

Definition 1.4. (Reidemeister torsion of a chain complex)

$$\mathbf{C}_* : \ C_m \xrightarrow{\partial_{m-1}} C_{m-1} \xrightarrow{\partial_{m-2}} \cdots \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_0} C_0$$

を体 F 上の finitely generated free chain complex とする.

C_i の basis $\mathbf{c}_i = (c_i^{(1)}, \cdots, c_i^{(p_i)}) \ (i = 0, \cdots, m)$ を固定し、$\mathbf{c} = (\mathbf{c}_0, \cdots, \mathbf{c}_m)$ とすると、これは $\mathbf{C}_*$ の basis である.

$$Z_i := \mathrm{Ker}\,(\partial_{i-1}), \ B_i := \mathrm{Im}\,(\partial_i), \ H_i := H_i(\mathbf{C}_*) = Z_i / B_i$$

とおく $(i = 0, \cdots, m)$. このとき、

$$C_i \cong Z_i \oplus B_{i-1} \cong B_i \oplus H_i \oplus B_{i-1}.$$

B_i の basis $\mathbf{b}_i = (b_i^{(1)}, \cdots, b_i^{(q_i)})$、$H_i$ の basis $\mathbf{h}_i = (h_i^{(1)}, \cdots, h_i^{(r_i)})$ を取る. $\mathbf{b}_{i-1}$ の ∂_{i-1} による lift を $\tilde{\mathbf{b}}_{i-1}$ と表し、$\mathbf{h}_i$ の自然な全射 $Z_i \to H_i$ による lift を $\tilde{\mathbf{h}}_i$ と表すと、上の同形対応により、$\mathbf{b}_i\tilde{\mathbf{h}}_i\tilde{\mathbf{b}}_{i-1}$ が C_i の basis を成す. このとき、

$$\tau(\mathbf{C}_*; \mathbf{c}) := \prod_{i=0}^{m} \left[\mathbf{b}_i\tilde{\mathbf{h}}_i\tilde{\mathbf{b}}_{i-1}/\mathbf{c}_i\right]^{(-1)^{i+1}}$$

を $\mathbf{C}_*$ の *Reidemeister torsion* と定義する. これは $F - \{0\}$ の元である. $\mathbf{c}$ が明らかなときは $\tau(\mathbf{C}_*)$ と表す.

$\tau(\mathbf{C}_*)$ を求める際に自由度を生む要素は以下である.

① $\mathbf{c}$ の取り方. ② $\mathbf{b}$ の取り方. ③ $\mathbf{h}$ の取り方.

④ $\tilde{\mathbf{b}}$ の取り方. ⑤ $\tilde{\mathbf{h}}$ の取り方.

どの要素がどれ程の自由度を生むのか明確にしておく.

[① について] $\mathbf{C}_*$ の別の basis を $\mathbf{c}'$ とすると、

$$\frac{\tau(\mathbf{C}_*; \mathbf{c})}{\tau(\mathbf{C}_*; \mathbf{c}')} = \prod_{i=0}^{m} \left[\mathbf{c}'_i/\mathbf{c}_i\right]^{(-1)^{i+1}}$$

で、これは $F - \{0\}$ のどの元も取り得る. だから、$\mathbf{c}$ の取り方の自由度を無闇に認めると invariant としての意味を成さない.

一方、他の要素は事情が違ってくる.

Lemma 1.5. *Let* $\mathbf{b}_i$ *and* $\mathbf{h}_i$ *be fixed bases of* B_i *and* H_i*, respectively. Then the equivalence class of* $\mathbf{b}_i\tilde{\mathbf{h}}_i\tilde{\mathbf{b}}_{i-1}$ *does not depend on lifts* $\tilde{\mathbf{h}}_i$ *and* $\tilde{\mathbf{b}}_{i-1}$.

Proof $p_i = \mathrm{rank}\,(C_i),\ q_i = \mathrm{rank}\,(B_i),\ r_i = \mathrm{rank}\,(H_i)$ とおく. $p_i = q_i + r_i + q_{i-1}$ が成立する.

$\tilde{\mathbf{h}}'_i,\ \tilde{\mathbf{b}}'_{i-1}$ を $\mathbf{h}_i,\ \mathbf{b}_{i-1}$ の別の lift とする.

$$\begin{aligned}
&\mathbf{b}_i = (b_i^{(1)}, \cdots, b_i^{(q_i)}), \\
&\tilde{\mathbf{h}}_i = (\tilde{h}_i^{(1)}, \cdots, \tilde{h}_i^{(r_i)}), && \tilde{\mathbf{h}}'_i = (\tilde{h}_i'^{(1)}, \cdots, \tilde{h}_i'^{(r_i)}), \\
&\tilde{\mathbf{b}}_{i-1} = (\tilde{b}_{i-1}^{(1)}, \cdots, \tilde{b}_{i-1}^{(q_{i-1})}), && \tilde{\mathbf{b}}'_{i-1} = (\tilde{b}_{i-1}'^{(1)}, \cdots, \tilde{b}_{i-1}'^{(q_{i-1})}), \\
&\mathbf{d}_i = \tilde{\mathbf{h}}'_i - \tilde{\mathbf{h}}_i = (d_i^{(1)}, \cdots, d_i^{(r_i)}), && \mathbf{e}_i = \tilde{\mathbf{b}}'_{i-1} - \tilde{\mathbf{b}}_{i-1} = (e_i^{(1)}, \cdots, e_i^{(q_{i-1})})
\end{aligned}$$

とおく. このとき、$d_i^{(j)} \in B_i,\ e_i^{(j)} \in Z_i \cong B_i \oplus H_i$. これより、

$$d_i^{(j)} = \sum_{k=1}^{q_i} s_{jk} b_i^{(k)}$$
$$e_i^{(j)} = \sum_{k=1}^{q_i} t_{jk} b_i^{(k)} + \sum_{l=1}^{r_i} u_{jl} \tilde{h}_i^{(l)}.$$

$$S = \begin{pmatrix} s_{jk} \end{pmatrix},\ T = \begin{pmatrix} t_{jk} \end{pmatrix},\ U = \begin{pmatrix} u_{jl} \end{pmatrix}$$

とおくと、

$$\left[\mathbf{b}_i \tilde{\mathbf{h}}_i' \tilde{\mathbf{b}}_{i-1}' / \mathbf{b}_i \tilde{\mathbf{h}}_i \tilde{\mathbf{b}}_{i-1}\right] = \det \begin{pmatrix} I_{r_i} & O & O \\ S & I_{q_i} & O \\ T & U & I_{r_{i-1}} \end{pmatrix} = 1$$

となり、示された. □

Lemma 1.6. *Let* $\mathbf{h}_i$ *be a fixed basis of* H_i*. Then the value* $\tau(\mathbf{C}_*; \mathbf{c})$ *does not depend on choices of* $\mathbf{b}_i$*'s.*

Proof $\mathbf{b}_i' = (b_i'^{(1)}, \cdots, b_i'^{(q_i)})$ を B_i の別の basis とする.

$$b_i'^{(j)} = \sum_{k=1}^{q_i} s_{jk,i} b_i^{(k)}$$
$$\tilde{b}_{i-1}^{(j)} = \sum_{k=1}^{q_i} t_{jk,i} b_i^{(k)} + \sum_{l=1}^{r_i} u_{jl,i} \tilde{h}_i^{(l)} + \sum_{m=1}^{q_{i-1}} s_{jm,i-1} \tilde{b}_{i-1}^{(m)}.$$

$$S_i = \begin{pmatrix} s_{jk,i} \end{pmatrix},\ T_i = \begin{pmatrix} t_{jk,i} \end{pmatrix},\ U_i = \begin{pmatrix} u_{jl,i} \end{pmatrix}$$

とおくと、S_i は $q_i \times q_i$-行列で、$S_{-1} = S_m = (\emptyset)$. T_i は $q_{i-1} \times r_i$-行列、U_i は $q_{i-1} \times q_i$-行列.

$$\prod_{i=0}^{m} \left[\mathbf{b}_i' \tilde{\mathbf{h}}_i \tilde{\mathbf{b}}_{i-1}' / \mathbf{b}_i \tilde{\mathbf{h}}_i \tilde{\mathbf{b}}_{i-1}\right]^{(-1)^{m+1}} = \prod_{i=0}^{m} \det \begin{pmatrix} S_i & O & O \\ O & I_{q_i} & O \\ T_i & U_i & S_{i-1} \end{pmatrix}^{(-1)^{m+1}}$$

$$= \det(S_{-1})^{-1} \det(S_m)^{(-1)^{m+1}} = \det(\emptyset)^{-1} \det(\emptyset)^{(-1)^{m+1}} = 1$$

となり、示された. □

以上のように、$\tau(\mathbf{C}_*; \mathbf{c})$ は $\mathbf{c}, \mathbf{h}$ の取り方による invariant である. 次節で空間からの Reidemeister torsion を定義するとき、$\mathbf{c}$ は自然なものに取ることができて、自由度を制限することができる. $\mathbf{h}$ も含めた Reidemeister torsion を利用した論文として、[18] を挙げておく.

以下、任意の i で $\mathrm{rank}\,(H_i) = 0$、つまり $\mathbf{C}_*$ が acyclic（$\mathbf{h} = \emptyset$）の場合のみを主に考えていく. $\mathbf{C}_*$ が non-acyclic のときは $\tau(\mathbf{C}_*) = 0$ と約束する.

$$\tau(\mathbf{C}_*;\mathbf{c}) := \begin{cases} \displaystyle\prod_{i=0}^{m} \left[\mathbf{b}_i\tilde{\mathbf{b}}_{i-1}/\mathbf{c}_i\right]^{(-1)^{i+1}} & (H_*(\mathbf{C}_*)=0), \\ 0 & (H_*(\mathbf{C}_*)\neq 0). \end{cases}$$

Example 1.7. (1) $\mathbf{C}_*: \ C_1 \xrightarrow{\partial_0} C_0$

体 F 上の chain complex として、C_1, C_0 の basis を $\mathbf{c}_1, \mathbf{c}_0$ とする.

$$\mathbf{C}_* \text{ が acyclic} \iff \partial_0 \text{ が isomorphism.}$$

∂_0 が isomorphism として、$\mathbf{c}_1 = (c_1^{(1)}, \cdots, c_1^{(r)}),\ \mathbf{c}_0 = (c_0^{(1)}, \cdots, c_0^{(r)})$ とおく.

$\partial_0(c_1^{(i)}) = \displaystyle\sum_{j=1}^{r} \alpha_{ij} c_0^{(j)}$ で ∂_0 は特徴付けられ、$A = \begin{pmatrix} \alpha_{ij} \end{pmatrix}$ は正則行列である.

$B_1 = 0,\ B_0 = C_0,\ B_{-1} = 0.$

$B_1 = 0,\ B_0 = C_0$ より、$\mathbf{b}_1 = \emptyset,\ \tilde{\mathbf{b}}_0 = \mathbf{c}_1,$

$\tilde{\mathbf{b}}_0 = \mathbf{c}_1,\ B_{-1} = 0$ より、$\mathbf{b}_0 = \partial_0(\mathbf{c}_1),\ \tilde{\mathbf{b}}_{-1} = \emptyset.$

$\tau(\mathbf{C}_*;\mathbf{c}) = \left[\mathbf{b}_0\tilde{\mathbf{b}}_{-1}/\mathbf{c}_0\right]^{(-1)^1} \left[\mathbf{b}_1\tilde{\mathbf{b}}_0/\mathbf{c}_1\right]^{(-1)^2} = \left[\partial_0(\mathbf{c}_1)/\mathbf{c}_0\right]^{-1} \left[\mathbf{c}_1/\mathbf{c}_1\right]$

$= (\det A)^{-1}.$

(2) $\mathbf{C}_*: \ C_2 \xrightarrow{\partial_1} C_1$

(1) と同様な設定の下、$\mathbf{C}_*$ が acyclic のとき、$\tau(\mathbf{C}_*;\mathbf{c}) = \det A.$

Multiplicity

Reidemeister torsion の計算法の唯一にして最大のテクニックと言ってもよい multiplicity について述べる.

$\mathbf{C}_*$ を体 F 上の finitely generated free chain complex、$\mathbf{C}'_*$ をその subcomplex、$\mathbf{C}''_* = \mathbf{C}_*/\mathbf{C}'_*$ を quotient chain complex とする.

$$\begin{array}{lccccccccc} \mathbf{C}_*: & C_m & \xrightarrow{\partial_{m-1}} & C_{m-1} & \xrightarrow{\partial_{m-2}} & \cdots & \xrightarrow{\partial_1} & C_1 & \xrightarrow{\partial_0} & C_0 \\ \mathbf{C}'_*: & C'_m & \xrightarrow{\partial'_{m-1}} & C'_{m-1} & \xrightarrow{\partial'_{m-2}} & \cdots & \xrightarrow{\partial'_1} & C'_1 & \xrightarrow{\partial'_0} & C'_0 \\ \mathbf{C}''_*: & C''_m & \xrightarrow{\partial''_{m-1}} & C''_{m-1} & \xrightarrow{\partial''_{m-2}} & \cdots & \xrightarrow{\partial''_1} & C''_1 & \xrightarrow{\partial''_0} & C''_0 \end{array}$$

$\mathbf{c}_i, \mathbf{c}'_i, \mathbf{c}''_i$ をそれぞれ C_i, C'_i, C''_i の basis で、2 つの写像

$$\iota : C'_i \hookrightarrow C_i, \quad p : C_i \to C''_i$$

に関して自然なものとする. つまり、$\tilde{\mathbf{c}}''_i$ を $\mathbf{c}''_i$ の p による lift とするとき、

$$\left[\iota(\mathbf{c}'_i)\tilde{\mathbf{c}}''_i/\mathbf{c}_i\right] = 1$$

が成立するものとする. 以下これを満たしているとする.

Lemma 1.8. *If two of* $\mathbf{C}_*, \mathbf{C}'_*$ *and* $\mathbf{C}''_*$ *are acyclic, then the rest is also acyclic.*

Proof 以下のような exact sequence がある.

$$\cdots \longrightarrow H_i(\mathbf{C}'_*) \longrightarrow H_i(\mathbf{C}_*) \longrightarrow H_i(\mathbf{C}''_*) \longrightarrow \cdots$$

例えば、$\mathbf{C}'_*, \mathbf{C}''_*$ が acyclic とすると、$H_i(\mathbf{C}'_*) = 0, H_i(\mathbf{C}''_*) = 0$ なので、$H_i(\mathbf{C}_*) = 0$ となる. つまり、$\mathbf{C}_*$ も acyclic である. 他の組合せでも同様である. □

Lemma 1.9. *If* $\mathbf{C}_*, \mathbf{C}'_*$ *and* $\mathbf{C}''_*$ *are acyclic, then there is the natural exact sequence induced by* ι *and* p*,*

$$0 \longrightarrow B'_i \xrightarrow{f} B_i \xrightarrow{g} B''_i \longrightarrow 0\ .$$

Proof f, g の well-defined 性と chain の exact 性を示す.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C'_{i+2} & \xrightarrow{\iota_{i+2}} & C_{i+2} & \xrightarrow{p_{i+2}} & C''_{i+2} & \longrightarrow & 0 \\
 & & {\scriptstyle \partial'_{i+1}}\downarrow & & {\scriptstyle \partial_{i+1}}\downarrow & & {\scriptstyle \partial''_{i+1}}\downarrow & & \\
0 & \longrightarrow & C'_{i+1} & \xrightarrow{\iota_{i+1}} & C_{i+1} & \xrightarrow{p_{i+1}} & C''_{i+1} & \longrightarrow & 0 \\
 & & {\scriptstyle \partial'_{i}}\downarrow & & {\scriptstyle \partial_{i}}\downarrow & & {\scriptstyle \partial''_{i}}\downarrow & & \\
0 & \longrightarrow & \mathrm{Im}\,(\partial'_i) & (\xrightarrow{f}) & \mathrm{Im}\,(\partial_i) & (\xrightarrow{g}) & \mathrm{Im}\,(\partial_i) & \longrightarrow & 0 \\
 & & \text{inclusion}\downarrow & & \text{inclusion}\downarrow & & \text{inclusion}\downarrow & & \\
0 & \longrightarrow & C'_i & \xrightarrow{\iota_i} & C_i & \xrightarrow{p_i} & C''_i & \longrightarrow & 0
\end{array}$$

Step 1. f の定義と、その単射性.

• x を $\mathrm{Im}\,(\partial'_i)$ の任意の元とすると、C'_{i+1} の元 a が存在して、$x = \partial'_i(a)$ とできる. このとき、$f(x) := \partial_i \circ \iota_{i+1}(a)$ と定義する.

• well-defined 性：

C'_{i+1} の元 b を、$x = \partial'_i(b)$ となるものとする. $\partial'_i(a-b) = 0$ より、$a-b \in \mathrm{Ker}\,(\partial'_i) = \mathrm{Im}\,(\partial'_{i+1})$. $a - b = \partial'_{i+1}(c)$ となる C'_{i+2} の元 c が存在する. $\partial_i \circ \iota_{i+1}(a-b) = \partial_i \circ \underline{\iota_{i+1} \circ \partial'_{i+1}}(c) = \underline{\partial_i \circ \partial_{i+1}} \circ \iota_{i+2}(c) = 0$ より、$\partial_i \circ \iota_{i+1}(a) = \partial_i \circ \iota_{i+1}(b)$.

• f は ι_i の制限なので、単射である.

Step 2. g の定義と、その全射性.

• y を $\mathrm{Im}\,(\partial_i)$ の任意の元とすると、C_{i+1} の元 α が存在して、$y = \partial_i(\alpha)$ とできる. このとき、$g(y) := \partial''_i \circ p_{i+1}(\alpha)$ と定義する.

• well-defined 性：

C_{i+1} の元 β を、$y = \partial_i(\beta)$ となるものとする. $\partial_i(\alpha-\beta) = 0$ より、$\alpha-\beta \in \mathrm{Ker}(\partial_i) = \mathrm{Im}\,(\partial_{i+1})$. $\alpha - \beta = \partial_{i+1}(\gamma)$ となる C_{i+2} の元 γ が存在する. $\partial''_i \circ p_{i+1}(\alpha - \beta) = \partial''_i \circ \underline{p_{i+1} \circ \partial_{i+1}}(\gamma) = \underline{\partial''_i \circ \partial''_{i+1}} \circ p_{i+2}(\gamma) = 0$ より、$\partial''_i \circ p_{i+1}(\alpha) = \partial''_i \circ p_{i+1}(\beta)$.

• $g \circ \partial_i = \partial''_i \circ p_{i+1}$. ∂''_i, p_{i+1} は全射なので、$g \circ \partial_i$ は全射. よって、g も全射.

Step 3. $\mathrm{Im}\,(f) = \mathrm{Ker}\,(g)$ であること.

$[\mathrm{Im}\,(f) \subset \mathrm{Ker}\,(g)]$

• $g \circ f(x) = \partial''_i \circ \underline{p_{i+1} \circ \iota_{i+1}}(a) = 0$ より、$\mathrm{Im}\,(f) \subset \mathrm{Ker}\,(g)$.

$[\mathrm{Im}\,(f) \supset \mathrm{Ker}\,(g)]$

• $y \in \mathrm{Ker}\,(g)$ とする. $\partial_i(\alpha) = y$ となる $\alpha \in C_{i+1}$ も取る. $g \circ \partial_i(\alpha) = \partial''_i \circ p_{i+1}(\alpha) = 0$ より、$p_{i+1}(\alpha) \in \mathrm{Ker}\,(\partial''_i) = \mathrm{Im}\,(\partial''_{i+1})$. $p_{i+1}(\alpha) = \partial''_{i+1}(z)$ となる $z \in C''_{i+2}$ が存在する. p_{i+2} は全射なので、$p_{i+2}(\beta) = z$ となる $\beta \in C_{i+2}$ が存在する.

• $\alpha' = \alpha - \partial_{i+1}(\beta)$ とおくと、$\partial_i(\alpha') = \partial_i(\alpha - \partial_{i+1}(\beta)) = \partial_i(\alpha) = y$.
$p_{i+1}(\alpha') = p_{i+1}(\alpha) - \underline{p_{i+1} \circ \partial_{i+1}}(\beta) = p_{i+1}(\alpha) - \underline{\partial''_{i+1} \circ p_{i+2}}(\beta)$.
$\partial''_{i+1} \circ \underline{p_{i+2}(\beta)} = \partial''_{i+1}(z) = p_{i+1}(\alpha)$ より、$p_{i+1}(\alpha') = p_{i+1}(\alpha) - p_{i+1}(\alpha) = 0$. これより、$\alpha' \in \mathrm{Ker}\,(p_{i+1}) = \mathrm{Im}\,(\iota_{i+1})$.

• $\alpha' = \iota_{i+1}(a')$ となる $a' \in C'_{i+1}$ が存在する. $x = \partial'_i(a') \in \mathrm{Im}\,(\partial'_i)$ とおくとき、$f(x) = \partial_i \circ \underline{\iota_{i+1}(a')} = \partial_i(\alpha') = y$ なので、$y \in \mathrm{Im}\,(f)$. □

Theorem 1.10. (Whitehead [30]; Turaev [25]) *If* $\mathbf{C}'_*$ *or* $\mathbf{C}''_*$ *is acyclic, then*

$$\tau(\mathbf{C}_*; \mathbf{c}) = \pm\tau(\mathbf{C}'_*; \mathbf{c}')\tau(\mathbf{C}''_*; \mathbf{c}'').$$

Proof C_i で考えるのが明らかなときは、$\iota(\mathbf{c}'_i), \tilde{\mathbf{c}}''_i$ を $\mathbf{c}'_i, \mathbf{c}''_i$ とも表すことにする. B_i, B'_i, B''_i の basis を $\mathbf{b}_i, \mathbf{b}'_i, \mathbf{b}''_i$ とする. Lemma 1.9 より $\left[\mathbf{b}_i/\mathbf{b}'_i\mathbf{b}''_i\right] = 1$ と取れる.

Lemma 1.8 より、$\mathbf{C}_*$ が non-acyclic であることと、$\mathbf{C}'_*, \mathbf{C}''_*$ の少なくとも一方が non-acyclic であることとは同値である. このとき、両辺 0 で等号成立.

以下、$\mathbf{C}_*$ が acyclic を仮定する. $\mathbf{C}'_*, \mathbf{C}''_*$ の一方が acyclic のとき、Lemma 1.8 より他方も acyclic. よって 3 つとも acyclic.

$$\tau(\mathbf{C}_*; \mathbf{c}) = \prod_{i=0}^{m} \left[\mathbf{b}_i\tilde{\mathbf{b}}_{i-1}/\mathbf{c}_i\right]^{(-1)^{i+1}} = \prod_{i=0}^{m} \left[\iota(\mathbf{b}'_i)\tilde{\mathbf{b}}''_i\iota(\tilde{\mathbf{b}}'_{i-1})\tilde{\tilde{\mathbf{b}}}''_i/\iota(\mathbf{c}'_i)\tilde{\mathbf{c}}''_i\right]^{(-1)^{i+1}}$$

$$= \pm\prod_{i=0}^{m} \left[\mathbf{b}'_i\tilde{\mathbf{b}}'_{i-1}/\mathbf{c}'_i\right]^{(-1)^{i+1}} \cdot \prod_{i=0}^{m} \left[\mathbf{b}''_i\tilde{\mathbf{b}}''_{i-1}/\mathbf{c}''_i\right]^{(-1)^{i+1}} = \pm\tau(\mathbf{C}'_*; \mathbf{c}')\tau(\mathbf{C}''_*; \mathbf{c}''). \quad \square$$

各 B'_i, B''_i の rank がわかれば、$\pm$ の特定まで可能である.

Changing coefficients

これまで chain complex $\mathbf{C}_*$ は体 F を係数としていたが、環 R を係数とする場合も考えるべきである. このときの環 R は Proposition 1.2 の直前にある条件 $(*)$ を満

たすものとする．特に環 R が integral domain のときは、$\mathbf{C}_*$ の Reidemeister torsion としては、$\mathbf{C}_*$ に商体 $Q(R)$ をテンソル積をしたものの Reidemeister torsion のこととして定義する．

$$\tau(\mathbf{C}_*) := \tau(\mathbf{C}_* \otimes Q(R)) \in Q(R)$$

一般の環 R を係数とするときは、integral domain R' への環準同形写像 $\varphi : R \to R'$ によって、$\mathbf{C}_* \otimes_R Q(R')$ の Reidemeister torsion のこととして定義する．

$$\tau^{\varphi}(\mathbf{C}_*) := \tau(\mathbf{C}_* \otimes_R Q(R'))$$

と表す．

Notation $\mathbf{C}_*^{\varphi} := \mathbf{C}_* \otimes_R Q(R'),\ H_*^{\varphi}(\mathbf{C}_*) := H_*(\mathbf{C}_*^{\varphi}),\ \tau^{\varphi}(\mathbf{C}_*) := \tau(\mathbf{C}_*^{\varphi})$.

Remark 1.11. (1) $\tau^{\varphi}(\mathbf{C}_*) \neq 0$ は $H_*^{\varphi}(\mathbf{C}_*) = 0$ と同値．

(2) Euler 数 $\chi(\mathbf{C}_*) \neq 0$ のとき、$\tau^{\varphi}(\mathbf{C}_*) = 0$.

Example 1.12. (1) $\mathbf{C}_* : \ C_2 \xrightarrow{\partial_1} C_1$

$G = \langle t \mid t^3 = 1 \rangle \cong \mathbf{Z}/3\mathbf{Z},\ R = \mathbf{Z}[G] = \mathbf{Z}[t, t^{-1}]/(t^3 - 1)$ とする．$\mathbf{C}_*$ を R 係数の chain complex とし、$\mathbf{c}_2 = (c_2^{(1)}),\ \mathbf{c}_1 = (c_1^{(1)}),\ \partial_1(c_2^{(1)}) = (t^2 + t + 1)c_1^{(1)}$ とする．このとき $\mathrm{Ker}\,(\partial_1) = (t - 1) \neq (0)$ より、$\mathbf{C}_*$ そのものは non-acyclic である．R は integral domain でないことも注意しておく．

ζ を 1 の原始 3 乗根として、$\varphi : \mathbf{Z}[G] \to \mathbf{Q}(\zeta)$ を $\varphi(1) = 1,\ \varphi(t) = \zeta$ から決まる環準同形、$\psi : \mathbf{Z}[G] \to \mathbf{Q}$ を $\psi(1) = \psi(t) = 1$ から決まる環準同形とする．このとき、$\mathbf{C}_*^{\varphi}$ は non-acyclic. $\mathbf{C}_*^{\psi}$ は acyclic で、$\tau^{\psi}(\mathbf{C}_*) = 3$ であることが確かめられる．

(2) $\mathbf{C}_* : \ C_3 \xrightarrow{\partial_2} C_2 \xrightarrow{\partial_1} C_1$

$G,\ R,\ \varphi,\ \psi$ は (1) と同じものとする．$\mathbf{C}_*$ は R 係数の chain complex で、$\mathbf{c}_3 = (c_3^{(1)}),\ \mathbf{c}_2 = (c_2^{(1)}, c_2^{(2)}),\ \mathbf{c}_1 = (c_1^{(1)}),\ \partial_2(c_3^{(1)}) = (t - 1)c_2^{(1)},\ \partial_1(c_2^{(1)}) = (t^2 + t + 1)c_1^{(1)},\ \partial_1(c_2^{(2)}) = (t^2 - 1)c_1^{(1)}$ とする．$\mathrm{Ker}\,(\partial_2) = (t^2 + t + 1) \neq (0)$ より、$\mathbf{C}_*$ そのものは non-acyclic である．

$\mathbf{C}_*^{\varphi}$ は acyclic で、$\tau^{\varphi}(\mathbf{C}_*) = \zeta + 1$. $\mathbf{C}_*^{\psi}$ は non-acyclic であることが確かめられる．

以上により、chain complex と環準同形の組合せによって acyclic 性は大きく変化することを注意しておく．

1.2. Reidemeister torsion of CW-complex

Definition 1.13. (Reidemeister torsion of a CW-complex)

X を finite CW-complex、$p : \hat{X} \to X$ を X の maximal abelian covering とする．このとき $\hat{X}$ には X から自然に誘導される CW-structure が入る．各 cell には p の

covering transformation group $H = H_1(X;\mathbf{Z})$ の元が作用するので、chain complex $\mathbf{C}_*(\hat{X})$ は $\mathbf{Z}[H]$ 上の finitely generated free chain complex と見なせる (Figure 2). 環準同形 $\varphi : \mathbf{Z}[H] \to F$ に対して、$\tau(\mathbf{C}_*(\hat{X}) \otimes_{\mathbf{Z}[H]} F)$ を X の（φ に付随した）*Reidemeister torsion* と定義する.

Notation $\mathbf{C}^{\varphi}_*(X) := \mathbf{C}_*(\hat{X}) \otimes_{\mathbf{Z}[H]} F,\ H^{\varphi}_*(X) := H_*(\mathbf{C}^{\varphi}_*)$,

$$\tau^{\varphi}(X) := \begin{cases} \tau(\mathbf{C}^{\varphi}_*(X)) & \in F - \{0\} \quad (H^{\varphi}_*(X) = 0), \\ 0 & \in F \quad (H^{\varphi}_*(X) \neq 0). \end{cases}$$

$\tau^{\varphi}(X)$ の自由度は、(1) $\hat{X}$ の basis の取り方と、(2) その並べ方による. (1) は $\varphi(h)$ 倍 ($h \in H$)、(2) は ± 1 倍の自由度を生む. 以上により、Reidemeister torsion の値 $\tau^{\varphi}(X)$ は、$\pm\varphi(h)$ 倍 ($h \in H$) の自由度をもって定義される.

CW-pair (X, Y) に対して、$\tau^{\varphi}(X, Y) := \tau(\mathbf{C}^{\varphi}_*(X, p^{-1}(Y)))$ を (X, Y) の Reidemeister torsion と定義する. 自由度も同様である.

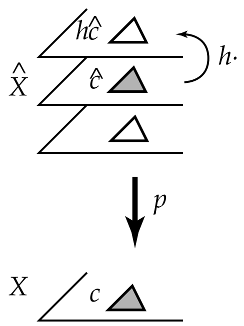

FIGURE 2. the maximal abelian covering of X

Theorem 1.14. [25] *Let (X, Y) be a finite CW-pair, $j : Y \hookrightarrow X$ the natural inclusion, and $\varphi : \mathbf{Z}[H_1(X;\mathbf{Z})] \to R$ a ring homomorphism. If $\tau^{\varphi\circ j}(Y) \neq 0$ or $\tau^{\varphi}(X, Y) \neq 0$, then*

$$\tau^{\varphi}(X) = \tau^{\varphi}(X, Y) \cdot \tau^{\varphi\circ j}(Y).$$

Proof $\mathbf{C}_* = \mathbf{C}^{\varphi}_*(X),\ \mathbf{C}'_* = \mathbf{C}^{\varphi\circ j}_*(Y),\ \mathbf{C}''_* = \mathbf{C}^{\varphi}_*(X, Y)$ とおくと、$\mathbf{C}''_* = \mathbf{C}_*/\mathbf{C}'_*$ である. Theorem 1.10 より、成立する. □

Theorem 1.14 も重要だが、以下の切除性質の方がより重要である.

Theorem 1.15. (excision) [25] *Let X be a finite CW-complex, X_1 and X_2 subcomplexes of X such that $X_1 \cup X_2 = X$, and $Y = X_1 \cap X_2$. Let $j : \mathbf{Z}[Y] \to \mathbf{Z}[X]$ and*

$j_i : \mathbf{Z}[X_i] \to \mathbf{Z}[X]$ $(j = 1, 2)$ *be homomorphisms induced by the natural inclusions, and* $\varphi : \mathbf{Z}[H_1(X;\mathbf{Z})] \to R$ *a ring homomorphism. If* $\tau^{\varphi\circ j}(Y) \neq 0$*, then*

$$\tau^{\varphi}(X) = \tau^{\varphi\circ j_1}(X_1) \cdot \tau^{\varphi\circ j_2}(X_2) \cdot (\tau^{\varphi\circ j}(Y))^{-1}.$$

Proof $\mathbf{C}'_* = \mathbf{C}^{\varphi\circ j}_*(Y)$, $\mathbf{C}_* = \mathbf{C}^{\varphi\circ j_1}_*(X_1) \oplus \mathbf{C}^{\varphi\circ j_2}_*(X_2) = \mathbf{C}^{\varphi\circ(j_1 \amalg j_2)}_*(X_1 \amalg X_2)$, $\mathbf{C}''_* = \mathbf{C}^{\varphi}_*(X)$ とおくと、

$$0 \to \mathbf{C}'_* \xrightarrow{j} \mathbf{C}_* \xrightarrow{j_1 - j_2} \mathbf{C}''_* \to 0$$

は Mayer-Vietoris exact sequence である．Theorem 1.10 より、成立する． □

まだこの時点では $\tau^{\varphi}(X)$ は X の CW-structure に付随した invariant である．じつは simple homotopy invariant をであること示すのが以下の定理である．

Theorem 1.16. [25] *Let* X *and* X' *be finite CW-complexes,* $f : X \to X'$ *a simple homotopy map,* $\varphi' : \mathbf{Z}[H_1(X)] \to R$ *a ring homomorphism, and* $\varphi = \varphi' \circ f_*$*. Then*

$$\tau^{\varphi}(X) = \tau^{\varphi'}(X').$$

Proof 任意の simple-homotopy は elementary simple homotopy の有限列で実現されるので、f が X から X' への elementary simple homotopy と仮定して示す．X の $(n-1)$-cell e を底面として、外部の点 x からの cone を張ることにより f を実現する (Figure 3).

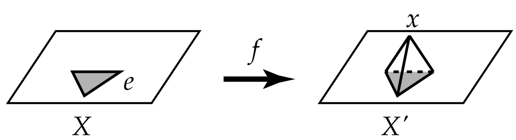

Figure 3. elementary simple homotopy

Theorem 1.14 より、$\tau^{\varphi'}(X') = \tau^{\varphi}(X) \cdot \tau^{\varphi'}(X', f(X))$. $e_n = x * e$ とおくと、$\tau^{\varphi'}(X', f(X)) = \tau^{\varphi'}(e_n, e)$ なので、$\tau^{\varphi'}(e_n, e) = 1$ を示せばよい．特に、$n = 3$ のときのみを示す．一般の場合も同様である．

$$C_3 \xrightarrow{\partial_2} C_2 \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_0} C_0$$

を CW-pair (e_3, e) から誘導される chain complex とする．

$\mathbf{c}_3 = (e_3)$, $\mathbf{c}_2 = (e_2^{(1)}, e_2^{(2)}, e_2^{(3)})$, $\mathbf{c}_1 = (e_1^{(1)}, e_1^{(2)}, e_1^{(3)})$, $\mathbf{c}_0 = (x)$ とするとき、

$\partial_2(e_3) = e_2^{(1)} + e_2^{(2)} + e_2^{(3)}$,

$\partial_1(e_2^{(1)}) = e_1^{(1)} - e_1^{(2)}$, $\partial_1(e_2^{(2)}) = e_1^{(2)} - e_1^{(3)}$, $\partial_1(e_2^{(3)}) = e_1^{(3)} - e_1^{(1)}$,

$\partial_0(e_1^{(1)}) = \partial_0(e_1^{(2)}) = \partial_0(e_1^{(3)}) = x.$

このとき、$\mathbf{b}_3 = \emptyset,\ \tilde{\mathbf{b}}_2 = (e_3),\ \mathbf{b}_2 = (\partial_2(e_3)),\ \tilde{\mathbf{b}}_1 = (e_2^{(2)}, e_2^{(3)}),$

$\mathbf{b}_1 = (\partial_1(e_2^{(2)}), \partial_1(e_2^{(3)})) = (e_1^{(2)} - e_1^{(3)}, e_1^{(3)} - e_1^{(1)}),\ \tilde{\mathbf{b}}_0 = (e_2^{(3)}),$

$\mathbf{b}_0 = (\partial_0(e_2^{(3)})),\ \tilde{\mathbf{b}}_{-1} = \emptyset$

と取ることができ、$\tau^{\varphi'}(e_n, e) = 1$ と求められる. □

CW-pair (X, Y) の Reidemeister torsion $\tau^\varphi(X, Y)$ も simple homotopy invariant であることは同様に示すことができる.

Corollary 1.17. [25] *The value* $\tau^\varphi(X)$ *is independent of subdivision.*

Proof 重心細分を simple homotopy で実現する. X を n 次元 CW-complex とするが、simplicial complex として示せば充分である. X の i 次元以下の cell 全体を X_i とする. 各 cell の内点を 1 点ずつ取っていき、X_n から順に細分していく.

Step 1. n-cell e と点 x との cone $x * e$ を取り、生じた $(n+1)$-cell を e 側からつぶす. これは、simple homotopy で実現する (Figure 4).

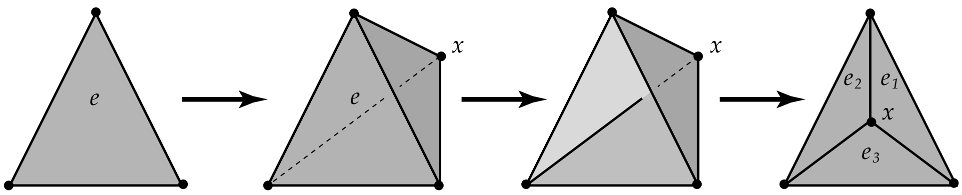

Figure 4. subdivision I

Step 2. 次の段階で Figure 5 の細分をしたい. X_1 の細分までたどり着いたとき、結果が重心細分になっている.

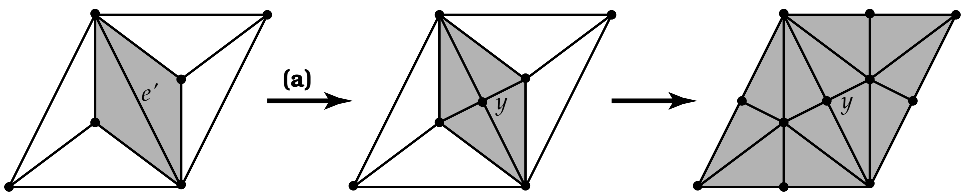

Figure 5. subdivision II

Step 3. (a) を詳しく見る.

$(n-1)$-cell e' を共有する cell 全体と、点 y との cone を取り、底面（この場合感覚的な用語）から次元の高い cell の順につぶしていく (Figure 6).

以降も同様にしていく. □

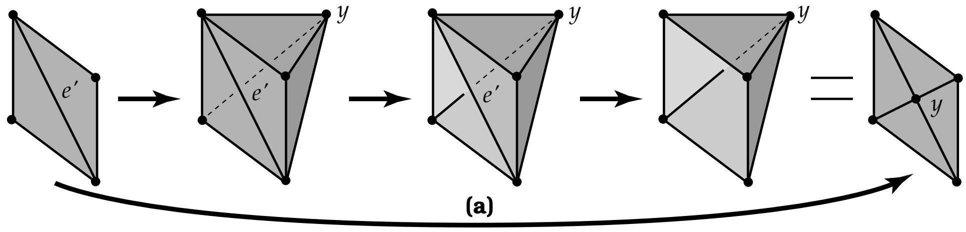

FIGURE 6. subdivision III

以上により、Reidemeister torsion が simple homotopy invariant であることと、細分で不変であることを示せた．しかしこれでも cell decomposition によっている．A. Chapman [2] は以下を示した．

Theorem 1.18. (Chapman [2]) *Any simple homotopy invariant is a topological invariant.*

この証明には、無限次元の立方体を用いる．筆者の理解を超えているので説明不可能である．とにかくこの定理により、Reidemeister torsion は topological invariant であることが保証された．

ここで、以降の議論に不可欠な 2 つの例を挙げておく．

Example 1.19. (1) $X = S^1$

$H_1(S^1) = \langle t \rangle \cong \mathbf{Z}$, $p : \hat{X} \to X$ を maximal abelian covering とする．S^1 を、1 つの 0-cell c_0、1 つの 1-cell c_1 に分割する．lift も同じ記号を使うことにする (Figure 7 (1)).

$$\mathbf{C}_* : \ C_1 \xrightarrow{\partial_0} C_0$$

を S^1 の分割から決まる $\hat{X}$ ($\cong \mathbf{R}^1$) の chain complex とする．

$\mathbf{c}_1 = (c_1)$, $\mathbf{c}_0 = (c_0)$, $\partial_0(c_1) = (t-1)c_0$ である．

$\mathbf{b}_1 = \emptyset$, $\tilde{\mathbf{b}}_0 = (c_1)$, $\mathbf{b}_0 = (\partial_0(c_1)) = ((t-1)c_0)$, $\tilde{\mathbf{b}}_{-1} = \emptyset$ より、

$$\tau(S^1) = (t-1)^{-1}.$$

(2) $X = S^1 \times S^1$

$H_1(S^1 \times S^1) = \langle g, h \rangle \cong \mathbf{Z} \oplus \mathbf{Z}$, $p : \hat{X} \to X$ を maximal abelian covering とする．$S^1 \times S^1$ を、1 つの 0-cell c_0、2 つの 1-cell c_1^1, c_1^2、1 つの 2-cell c_2 に分割する．lift も同じ記号を使うことにする (Figure 7 (2)).

$$\mathbf{C}_* : \ C_2 \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_0} C_0$$

を $S^1 \times S^1$ の分割から決まる $\hat{X}$ ($\cong \mathbf{R}^2$) の chain complex とする．

$\mathbf{c}_2 = (c_2),\ \mathbf{c}_1 = (c_1^1, c_1^2)\mathbf{c}_0 = (c_0),$
$\partial_1(c_2) = (1-h)c_1^1 + (g-1)c_1^2,$
$\partial_0(c_1^1) = (g-1)c_0,\ \partial_0(c_1^2) = (h-1)c_0$ である.
$\mathbf{b}_2 = \emptyset,\ \tilde{\mathbf{b}}_1 = (c_2),\ \mathbf{b}_1 = (\partial_1(c_2)) = ((1-h)c_1^1 + (g-1)c_1^2),\ \tilde{\mathbf{b}}_0 = (c_1^1),$
$\mathbf{b}_0 = (\partial_0(c_1^1)) = ((g-1)c_0),\ \tilde{\mathbf{b}}_{-1} = \emptyset$ より、

$$\tau(S^1 \times S^1) = -1.$$

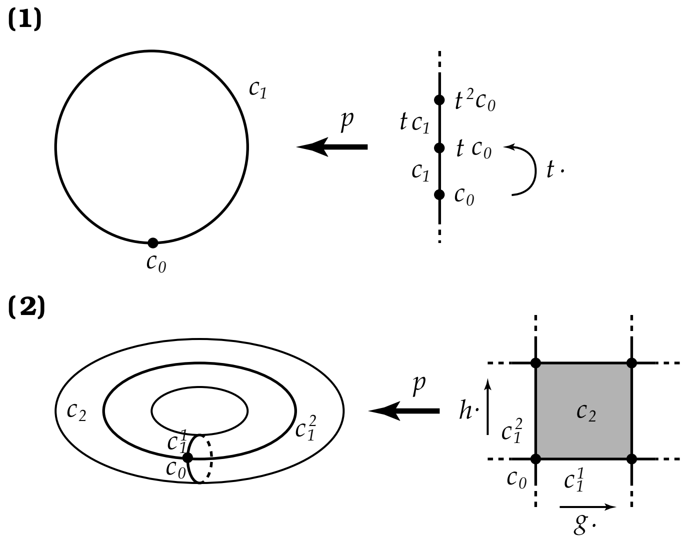

FIGURE 7. the maximal abelian coverings of S^1 and $S^1 \times S^1$

Proposition 1.20. (1) *Let t be a generator of $H_1(S^1)$, and $\varphi : \mathbf{Z}[t, t^{-1}] \to R$ a ring homomorphism. Then $\mathbf{C}_*^{\varphi}(S^1)$ is acyclic if and only if $\varphi(t) - 1$ is not zero divisor in R. If $\mathbf{C}_*^{\varphi}(S^1)$ is acyclic, then $\tau^{\varphi}(S^1) = (\varphi(t) - 1)^{-1}$.*

(2) *Let t_1 and t_2 be generators of $H_1(S^1 \times S^1)$, and $\varphi : \mathbf{Z}[t_1^{\pm 1}, t_2^{\pm 1}] \to R$ a ring homomorphism. Then $\mathbf{C}_*^{\varphi}(S^1 \times S^1)$ is acyclic if and only if $\varphi(t_1) - 1$ or $\varphi(t_2) - 1$ is not zero divisor in R. If $\mathbf{C}_*^{\varphi}(S^1 \times S^1)$ is acyclic, then $\tau^{\varphi}(S^1 \times S^1) = 1$.*

Example 1.21. (1) $\varphi : \mathbf{Z}[t, t^{-1}] \to \mathbf{Z}[s, s^{-1}]/(s^n - 1)$ を $\varphi(t) = s$ で決まる環準同形とする. $(s-1)(s^{n-1} + \cdots + s + 1) = s^n - 1 = 0$ より、$\varphi(t) - 1 = s - 1$ は zero divisor である.

(2) $\varphi : \mathbf{Z}[t, t^{-1}] \to \mathbf{Q}(\zeta)$ を $\varphi(t) = \zeta$ で決まる環準同形とする. ここで、ζ は 1 の原始 n 乗根である. $\varphi(t) - 1 = \zeta - 1$ は zero divisor ではない.

(3) $\varphi : \mathbf{Z}[t,t^{-1}]/(t^n-1) \to \mathbf{Q}(\zeta)$ を $\varphi(t)=\zeta$ で決まる環準同形とする．ここで、ζ は 1 の原始 n 乗根である．$\varphi(t)-1=\zeta-1$ は zero divisor ではない．

(4) $\psi_d : \mathbf{Z}[t,t^{-1}]/(t^n-1) \to \mathbf{Q}(\zeta)$ を $\psi_d(t)=\zeta$ で決まる環準同形とする． ここで、ζ は 1 の原始 d 乗根である．d (≥ 2) が n の約数のとき ψ_d は well-defined で、$\psi_d(t)-1=\zeta-1$ は zero divisor ではない．

1.3. Milnor torsion and the Alexander polynomial

この節は基本的に証明を与えない．証明は Turaev [25], [26], [27] を参照のこと．

Definition 1.22. (Milnor torsion)

X を finite CW-complex とし、$H := H_1(X),\ G := H/\mathrm{Tor}\,(H)$ とおく． $\mathrm{pr} : \mathbf{Z}[H] \to \mathbf{Z}[G]$ を自然な全射とする．このとき、$\tau^{\mathrm{pr}}\,(X)$ を *Milnor torsion* と定義する．

$\mathbf{Z}[G]$ は integral domain である．

Definition 1.23. (Fox derivative)

$F := \langle x_1, \cdots, x_m\rangle$ を $x_1, \cdots, x_m$ で生成する free group とする．任意の $r \in F$ は $\mathbf{Z}[F]$ の中で

$$r = 1 + \sum_{j=1}^{m} f_j(x_j-1) \qquad (f_j \in \mathbf{Z}[F])$$

と一意的に表される．このときの各 f_j を

$$f_j = \frac{\partial r}{\partial x_j}$$

と表す．そして、これを $\mathbf{Z}[F]$ に拡張したものを *Fox derivative* と定義する．つまり、

$$\frac{\partial}{\partial x_j} : \mathbf{Z}[F] \to \mathbf{Z}[F]$$

を

(1) $r \in F$ のとき $\left(\dfrac{\partial}{\partial x_j}\right)(r) = \dfrac{\partial r}{\partial x_j}$,

(2) $a, b \in \mathbf{Z},\ u, v \in \mathbf{Z}[F]$ のとき $\left(\dfrac{\partial}{\partial x_j}\right)(au+bv) = a\dfrac{\partial u}{\partial x_j} + b\dfrac{\partial v}{\partial x_j}$

で定める．このとき、以下が成立する．

(i) $\dfrac{\partial c}{\partial x_j} = 0 \qquad (c \in \mathbf{Z})$,

(ii) $\dfrac{\partial x_i}{\partial x_j} = \delta_{ij} \qquad$ (δ_{ij} は Kronecker の delta),

(iii) $\dfrac{\partial (uv)}{\partial x_j} = u\dfrac{\partial v}{\partial x_j} + v(1, \cdots, 1)\dfrac{\partial u}{\partial x_j} \qquad (u, v \in \mathbf{Z}[F])$,

(iv) $\dfrac{\partial r^{-1}}{\partial x_j} = -r^{-1}\dfrac{\partial r}{\partial x_j} \qquad (r \in F)$.

Definition 1.24. (Elementary ideal, Alexander polynomial)

$F = \langle x_1, \cdots, x_m \rangle$ を free group、$\pi = \langle x_1, \cdots, x_m \mid r_1, r_2, \cdots \rangle$ を finitely generated group とする．$p : \mathbf{Z}[F] \to \mathbf{Z}[\pi]$ を natural projection、$\alpha : \mathbf{Z}[\pi] \to \mathbf{Z}[H]$ を abelianization が誘導する homomorphism、$\eta = \alpha \circ p$ とする．このとき、

$$A(\pi) := \left(\ \eta\left(\frac{\partial r_i}{\partial x_j}\right) \ \right)_{i=1,\cdots;\ j=1,\cdots,m}$$

を π の *Alexander matrix* という．

$A(\pi)$ の $(m-1)$-小行列式で生成される $\mathbf{Z}[H]$ の ideal $E(\pi)$ を π の *elementary ideal* といい、

$$\Delta(\pi) := \gcd(\mathrm{pr}\,(E(\pi)))$$

を π の *Alexander polynomial* という．各々π の presentation によらず一意的に決まる．ただし、$\Delta(\pi)$ は $\pm\mathrm{pr}\,(h)$ $(h \in H)$ 倍の自由度がある．

位相空間 X の *Alexander-Fox polynomial* は、

$$\Delta(X) := \Delta(\pi_1(X))$$

と定義する．X が homology 3-sphere Σ 内の n 成分 link L のとき、各 meridian が代表する H の元を $t_1, \cdots, t_n$ として、

$$\Delta_L(t_1, \cdots, t_n) := \Delta(\overline{\Sigma - N(L)})$$

を L の *Alexander polynomial* とする．

Example 1.25. (1) $\pi = \mathbf{Z} = \langle x \mid \cdot \rangle = \langle x, y \mid r = y \rangle$

$H = \langle t \rangle \cong \mathbf{Z}$. $\dfrac{\partial r}{\partial x} = 0,\ \dfrac{\partial r}{\partial y} = 1$ より、$A(\mathbf{Z}) = \begin{pmatrix} 0 & 1 \end{pmatrix}$.

$E(\mathbf{Z}) = (1) = \mathbf{Z}[t, t^{-1}]$ より、$\Delta(\mathbf{Z}) = 1$.

これは trivial knot の Alexander polynomial が 1 であることを示す．

(2) $\pi = \langle x, y \mid r = x^p y^q \rangle \qquad (p,q) = 1;\ p, q \geq 2$

$H = \langle t \rangle \cong \mathbf{Z}$ で、$\eta(x) = t^q,\ \eta(y) = t^{-p}$ と対応する．

$\dfrac{\partial r}{\partial x} = x^{p-1} + \cdots + x + 1 = \dfrac{x^p - 1}{x - 1}$,

$\dfrac{\partial r}{\partial y} = x^p(y^{q-1} + \cdots + y + 1) = x^p\dfrac{y^q - 1}{y - 1}$.

$$E(\pi) = \left\langle \frac{t^{pq} - 1}{t^q - 1},\ t^{pq}\frac{t^{-pq} - 1}{t^{-p} - 1} \right\rangle, \qquad \Delta(\pi) = \frac{(t^{pq} - 1)(t - 1)}{(t^p - 1)(t^q - 1)}.$$

これは (p,q)-torus knot の Alexander polynomial を表す．

Definition 1.26. (Order of a module)

K を commutative ring、H を finitely generated K-module とする．このとき、$f: K^m \to K^n$ K-homomorphism で、$\mathrm{Coker}\,(f) \cong H$、$n$ が有限であるものが存在する．m は無限かも知れないことに注意する．

$$K^m \quad \xrightarrow{\ f\ } \quad K^n \to H \to 0 \ , \quad \text{exact}$$

K が Noetherian ならば m を有限に取れる．

A を f の $m \times n$-presentation matrix、$E(H)$ を A の n 次小行列式で生成される K の ideal とする．これは H の presentation によらず一意的に決まる．K が unique factorization domain (UFD) のとき、

$$\mathrm{ord}\,(H) := \gcd(E(H))$$

と定義する．H の *order* という．これは $\pm u$ (u は H の unit) 倍の自由度で決まる．

Example 1.27. $K = \mathbf{Z},\ H = \mathbf{Z}/a\mathbf{Z} \oplus \mathbf{Z}/b\mathbf{Z} \quad (a, b > 0)$

H の presentation matrix は $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ なので、$\mathrm{ord}\,(H) = ab$.

X を finite CW-complex、$H := H_1(X),\ G := H/\mathrm{Tor}\,(H),\ \mathrm{pr}: \mathbf{Z}[H] \to \mathbf{Z}[G]$ を natural projection とする．$\mathbf{Z}[G]$ は $\mathbf{Z}$ 上の有限次 Laurent 多項式環なので、Noetherian UFD である．よって、finitely generated $\mathbf{Z}[G]$-module M に対して $\mathrm{ord}\,(M)$ は well-defined である．

Definition 1.28. (Alexander function)

$q: \tilde{X} \to X$ を maximal free abelian covering で、$\mathbf{C}_*^{\mathrm{pr}}(X) = \mathbf{C}_*(\tilde{X})$ は finitely generated $\mathbf{Z}[G]$-chain complex である．このとき、

$$A(X) := \prod_{i=0}^{m} [\mathrm{ord}\,(H_i^{\mathrm{pr}}(X))]^{(-1)^{i+1}} \in Q(\mathbf{Z}[G])$$

を X の *Alexander function* と定義する．これは $\pm g$ ($g \in G$) 倍の自由度で決まる．$\mathbf{C}_*^{\mathrm{pr}}(X)$ が non-acyclic であれば、$A(X) = 0$ とする．また、finite CW-pair (X, Y) に対して、$A(X, Y)$ も同様に定義できる．

Example 1.29. (1) $X = S^1$

$H_1^{\mathrm{pr}}(X) = 0,\ H_0^{\mathrm{pr}}(X) = \mathbf{Z}[t, t^{-1}]/(t-1)$.

$\mathrm{ord}\,(H_1^{\mathrm{pr}}(X)) = 1,\ \mathrm{ord}\,(H_0^{\mathrm{pr}}(X)) = t - 1. \quad A(S^1) = (t-1)^{-1}$.

(2) $X = S^1 \times S^1$

$H_2^{\mathrm{pr}}(X) = 0,\ H_1^{\mathrm{pr}}(X) = 0,\ H_0^{\mathrm{pr}}(X) = \mathbf{Z}[t_1^{\pm 1}, t_2^{\pm 1}]/(t_1 - 1, t_2 - 1)$.

$\mathrm{ord}\,(H_2^{\mathrm{pr}}(X)) = \mathrm{ord}\,(H_1^{\mathrm{pr}}(X)) = 1,\ \mathrm{ord}\,(H_0^{\mathrm{pr}}(X)) = \gcd(t_1 - 1, t_2 - 1) = 1$.

$A(S^1 \times S^1) = 1.$

以下の定理は重要である.

Theorem 1.30. (Turaev [25]) *Let* (X, Y) *be a finite CW-pair. Then*

$$\tau^{\mathrm{pr}}\,(X, Y) = A(X, Y).$$

これにより、Alexander function が Milnor torsion の第 2 の定義と見なすことができる. これの証明には、*matrix* τ*-chain* という道具を用いる. いかにもこれの証明のためだけに作られた道具で、他に応用はなさそうだが、なかなか味わいのある証明である. 線形代数の延長と言ってもよい.

次は、3-dimensional manifold において Milnor torsion と Alexander polynomial の関係を述べた重要な定理である.

Theorem 1.31. (Turaev [25], [27]) *Let* M *be a connected compact 3-dimensional manifold with* $\chi(M) = 0$*, and* $H = H_1(M)$*. If* $\mathrm{rank}(H) \geq 2$*, then* $\tau^{\mathrm{pr}}\,(M) = \Delta(M)$*. If* $\mathrm{rank}\,(H) = 1$ *and* $H/\mathrm{Tor}\,(H) = \langle t \rangle \cong \mathbf{Z}$*, then*

$$\tau^{\mathrm{pr}}\,(M) = \begin{cases} \Delta(M) \cdot (t-1)^{-1} & (\partial M \neq \emptyset \text{ or } w_1(\mathrm{Tor}\,(H)) \neq 1), \\ \Delta(M) \cdot (t-1)^{-2} & (\partial M = \emptyset, \text{ and } w_1(H) = 1), \\ \Delta(M) \cdot (t^2-1)^{-1} & (\partial M = \emptyset, w_1(\mathrm{Tor}\,(H)) = 1, \text{ and } w_1(H) \neq 1). \end{cases}$$

これの証明法は、多様体の Poincaré duality を用いる. 以下、状況を絞っていく.

Corollary 1.32. *Let* Σ *be a homology* 3*-sphere,* K *a knot in* Σ*, and* $H_1(\overline{\Sigma - N(K)}) = \langle t \rangle \cong \mathbf{Z}$*. Then*

$$\tau(\overline{\Sigma - N(K)}) = \Delta_K(t)(t-1)^{-1}.$$

Proof Theorem 1.31 で、$\mathrm{rank}\,(H) = 1,\ \partial M \neq \emptyset$ の場合である. □

Notation (1) Homology 3-sphere Σ 内の knot K に沿って p/q-surgery した結果の多様体を $\Sigma(K; p/q)$ と表す.

(2) d (≥ 2) を p の約数とし、ζ を 1 の原始 d 乗根とする. このとき、$\psi_d : \mathbf{Z}[t, t^{-1}]/(t^p - 1) \to \mathbf{Q}(\zeta)$ ring homomorphism を $\psi_d(t) = \zeta$ から導かれるものとする.

Corollary 1.33. (Turaev [24], [25], [26]) *Let* p *and* q *be integers satisfying* $(p, q) = 1, |p| \geq 2$ *and* $q \neq 0$*,* d (≥ 2) *a divisor of* p*, and* ζ *a primitive* d*-th root of unity. Then*

$$\tau^{\psi_d}(\Sigma(K; p/q)) = \Delta_K(\zeta)(\zeta - 1)^{-1}(\zeta^{\bar{q}} - 1)^{-1},$$

where $q\bar{q} \equiv 1 \pmod p$.

Proof Theorem 1.15 (excision), Proposition 1.20 と Corollary 1.32 より導かれる. □

1 の p 乗根だけを調べるのではなく、あらゆる p の約数 d 乗根で調べることが後に効いてくる (Main Theorem 2 の証明を参照). 特に K を trivial knot とすると以下が導かれる.

Theorem 1.34. (Reidemeister [20]; Franz [6]) *Let* $L(p,q)$ *be the* (p,q)*-lens space. Then*

$$\tau^{\psi_d}(L(p,q)) = (\zeta - 1)^{-1}(\zeta^{\bar{q}} - 1)^{-1},$$

where $q\bar{q} \equiv 1 \pmod p$.

後で出てくる Franz の定理（Theorem 2.4）を用いることにより、lens space の分類定理を導くことができる.

Theorem 1.35. (Reidemeister [20]; Franz [6]; Brody [1]; Przytycki-Yasuhara [19]) *Two lens spaces* $L(p,q)$ *and* $L(p',q')$ *are homeomorphic if and only if* (1) $p = p'$, *and* (2) $q \equiv \pm q' \pmod p$ *or* $qq' \equiv \pm 1 \pmod p$.

Reidemeister torsion は少なくとも lens space に対しては完全な invariant である. 他の homology lens space に対してどれ程効力があるかを調べるのがこの稿のテーマである.

1.4. Examples

いくつか計算例を挙げる.

Example 1.36. (1) $X = S^1 \times S^2$

$H_1(S^1 \times S^2) = \langle t \rangle \cong \mathbf{Z}$. $X = S^1 \times S^2$ を 2 つの solid torus T_1, T_2 ($\cong S^1 \times D^2$) の和に分割する (Figure 8).

$H_1(T_i) = \langle t_i \rangle \cong \mathbf{Z}$ $(i = 1,2)$, $T_1 \cap T_2 = \partial T_1 = \partial T_2 \cong S^1 \times S^1$.

$T_1 \cap T_2 \hookrightarrow T_1$, $T_1 \cap T_2 \hookrightarrow T_2$ を介して、$H_1(X)$ で $t_1 = t_2 = t$ の関係式が入る.

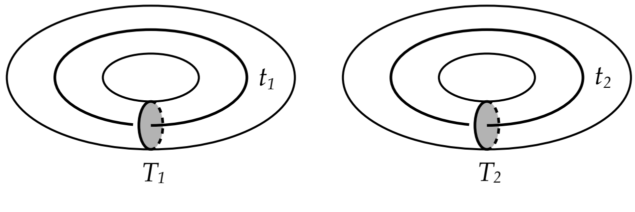

FIGURE 8. decomposition of $S^1 \times S^2$

$$\tau(S^1 \times S^2) = (t_1 - 1)^{-1}(t_2 - 1)^{-1} \cdot 1^{-1}|_{t_1=t_2=t} = (t-1)^{-2}.$$

(2) $X = S^1 \times S^n$ $(n \geq 2)$

$H_1(S^1 \times S^n) = \langle t \rangle \cong \mathbf{Z}$.

(1) と同様にするが、n についての帰納法で以下の結果を示す方法がよい.

$$\tau(S^1 \times S^n) = \begin{cases} (t-1)^{-2} & (n \text{ は偶数}) \\ 1 & (n \text{ は奇数}) \end{cases}$$

結果的に、$n = 1$ の場合でも正しくなっている.

(3) $X = S^1 \times B_n$ (B_n は、n 個の 1-cell、1 個の 0-cell よりなるブーケ. $n \geq 2$)

$H_1(S^1 \times B_n) = H_1(S^1) \oplus H_1(B_n) = \langle t \rangle \oplus \langle t_1, \cdots, t_n \rangle = \langle t, t_1, \cdots, t_n \rangle \cong \mathbf{Z}^{n+1}$. t は $H_1(S^1)$ の生成元、t_i $(i = 1, \cdots, n)$ は各 B_n の 1-cell に対応する元とする. $S^1 \times B_n$ は、n 個の torus を longitude に沿って接着した空間と見なせる.

$$\tau(S^1 \times B_n) = (t-1)^{n-1}.$$

これも結果的に、$n = 1$ の場合でも正しくなっている.

(4) $X = S^1 \times Y$

(2), (3) より、

$$\tau(S^1 \times Y) = (t-1)^{-\chi(Y)}.$$

ここで $\chi(Y)$ は、Y の Euler number である.

証明は、Y の cell 分割をして、Y の cell を e とするとき、$S^1 \times e$ を低い次元の順に接着していく手法である. その際、偶数と奇数で状況が異なることに注意.

これより、S^3 内の fibred knot complement が trivial bundle になるのは、trivial knot complement のときのみである.

(5) $X =$Klein bottle

$H_1(X) \cong \mathbf{Z} \oplus \mathbf{Z}_2,\ H_1(X)/\mathrm{Tor}\,(H_1(X)) = \langle t \rangle \cong \mathbf{Z}$.

torus と同様な分割だが、$\mathbf{Z}$-covering であることに注意.

$$\tau(X) = 1 \cdot (t+1) \cdot (t+1)^{-1} = 1.$$

(6) $X = S^1$ 上の B_n-bundle で、$\mathrm{rank}\,(H_1(X)) = 1$ となるもの.

$f : B_n \to B_n$ を monodromy map とする. (Figure 9)

$f_* : H_1(B_n) \to H_1(B_n)$ は isomorphism になる．自明な basis により f_* を表現したものを A とする．このとき、$\det(A) = \pm 1$.

I を単位行列として、$r = \mathrm{rank}\,(A - I)$ とおくと、$H_1(X)/\mathrm{Tor}\,(H_1(X)) \cong \mathbf{Z}^{n+1-r}$.

$\mathrm{rank}\,(H_1(X)) = 1 \iff r = n$、である．

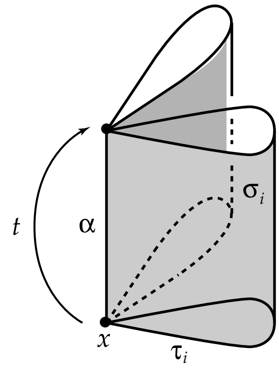

FIGURE 9

2-cell を σ_i、1-cell を τ_i, α、0-cell を x とする．
$\partial(\sigma_i) = tf_*(\tau_i) - \tau_i$,
$\partial(\tau_i) = 0,\ \partial(\alpha) = (t-1)x$.

$$\tau(X) = (t-1)^{-1} \det(tA - I).$$

特にこれより、fibred knot の最高次の係数が ± 1 であることと、Alexander polynomial の次数の 1/2 が、minimal genus と一致することがわかる．

(7) $X = (p, q)$-torus knot complement
$H_1(X) = \langle t \rangle \cong \mathbf{Z}$.

S^3 を 2 つの solid torus T_1, T_2 の和に分割する．(p, q)-torus knot K は、$\partial T_1 = \partial T_2$ の上に乗っていると見る．このとき K の近傍 $N(K)$ は、T_1, T_2 を削る．その削られた結果も solid torus で、それらを X_1, X_2 とする．$X = X_1 \cup X_2$ で、$X_1 \cap X_2$ は annulus である．

X_1 の core を l_1、meridean を m_1、
X_2 の core を l_2、meridean を m_2 とすると、
$H_1(X_1 \cap X_2) \cong \mathbf{Z}$ の生成元が、$[l_1]^p[m_1]^q$ または $[l_2]^q[m_2]^p$ と表される．
$[m_1] = 1,\ [m_2] = 1$ である．$[l_1] = t_1,\ [l_2] = t_2$ とおく．

$\tau(X_1) = (t_1 - 1)^{-1},\ \tau(X_2) = (t_2 - 1)^{-1}$.

$H_1(X)$ では、$t_1^p = t_2^q$ である.
$t_1 = t^q,\ t_2 = t^p$ とすると、$t_1^p = t_2^q = t^{pq}$ が成立.
$\tau(X_1) = (t^q - 1)^{-1},\ \tau(X_2) = (t^p - 1)^{-1},\ \tau(X_1 \cap X_2) = (t^{pq} - 1)^{-1}$.

$$\tau(X) = (t^p - 1)^{-1}(t^q - 1)^{-1}(t^{pq} - 1).$$

これより、(p, q)-torus knot の Alexander polynomial は

$$\Delta(X) = \frac{(t^{pq} - 1)(t - 1)}{(t^p - 1)(t^q - 1)}.$$

2. Rational surgery along a torus knot

2.1. Moser's Theorem

Main Theorem 1 の動機となった定理を述べる. S^3 内の torus knot に沿う rational surgery の結果の多様体を分類したものである.

Theorem 2.1. (Moser [16]; Gordon [8]; Shimozawa [23]) *Let $K_{r,s}$ be the (r, s)-torus knot in S^3, and $M = S^3(K_{r,s}; p/q)$ the result of p/q-surgery along $K_{r,s}$ where $|p|, |r|, |s| \geq 2$ and $q \neq 0$. Then there are three cases* :

(1) *If $|p - qrs| \neq 0$, then M is a Seifert fibered space with three singular fibers of multiplicities $|r|, |s|$ and $|p - qrs|$. In particular,*

(2) *if $|p - qrs| = 1$, then M is the lens space $L(p, qr^2)$* (Figure 1).

(3) *If $|p - qrs| = 0$ $(p/q = rs)$, then M is the connected sum of two lens spaces, $L(r, s)\sharp L(s, r)$.*

Reidemeister torsion を用いてこの結果にどれ程迫れるのか探っていく. Figure 1 は、$(2, 3)$-torus knot (trefoil) $K_{2,3}$ の 5-surgery の結果が lens space $L(5, 1)$ になることを表している (cf. [21], [23]). これらの Reidemeister torsion は一致するはずである. $M = S^3(K_{2,3}; 5)$、ζ を 1 の原始 5 乗根として Reidemeister torsion を計算する. Corollary 1.33, Theorem 1.34 より、

$$\tau^{\psi_5}(M) = (\zeta^2 - \zeta + 1)(\zeta - 1)^{-2}, \quad \tau^{\psi_5}(L(5, 1)) = (\zeta - 1)^{-2}.$$

これらの式は見かけ上違うものに見える. Reidemeister torsion は $\pm\zeta^m$ 倍の違いで定義されるので、

$$\zeta^2 - \zeta + 1 = \pm\zeta^m$$

が成り立つのではないか、と思うかも知れない. しかし、

$$|\zeta^2 - \zeta + 1| = |1 - 2\cos(2\pi k/5)| \neq |\pm\zeta^m| = 1$$

より、それは否定される．それではどうすれば同一視できるのだろうか？実は、(p,q)-torus knot の Alexander polynomial の公式までさかのぼると次の変形ができる．($\zeta^5=1$ も使う.)

$$\tau^{\psi_5}(M) = (\zeta^2-\zeta+1)(\zeta-1)^{-2} = \frac{(\zeta^6-1)(\zeta-1)}{(\zeta^3-1)(\zeta^2-1)}(\zeta-1)^{-2}$$

$$= \frac{(\zeta-1)(\zeta-1)}{(\zeta^{-2}-1)(\zeta^2-1)}(\zeta-1)^{-2} = -\zeta^2(\zeta^2-1)^{-2}.$$

はじめの $-\zeta^2$ はともかく、ζ を ζ^2 に置きかえた形で同一視できる．つまり、ψ_5 の取り方にも自由度があるのを見過ごしていたのである．

$$\psi_5 : \mathbf{Z}[t,t^{-1}]/(t^5-1) \to \mathbf{Q}(\zeta)$$

において、$\psi_5(t)=\zeta$ と定義したが、$\psi_5(t)=\zeta^2$ としても本質は変わらないのである．$\sigma:\zeta\to\zeta^2$ は Galois 群 $\mathrm{Gal}\,(\mathbf{Q}(\zeta)/\mathbf{Q})$ に属する元で、一般の場合にもこの自由度を考慮すべきである．以後、この "隠れた" 自由度をいかに捕まえるか、あるいは回避するかに心を砕いていく．

2.2. Main Theorem 1

Reidemeister torsion の値が一致した 2 つの空間が本当に同相か？という所まではこの稿では扱わない．ひとまず値が同じかどうかに関心を向ける．

Definition 2.2. (Lens space type)
ζ を 1 の原始 p 乗根とする．このとき、$a\in\mathbf{Q}(\zeta)$ が *lens space type* であるとは、

$$a = \pm\zeta^m(\zeta^i-1)^{-1}(\zeta^j-1)^{-1} \qquad m,i,j\in\mathbf{Z};(i,p)=1,\ (j,p)=1$$

の形に表されるときをいう．

M は orientable closed 3-manifold で、$H_1(M)\cong\mathbf{Z}/p\mathbf{Z}$ $(|p|\geq 2)$ であるとする．つまり、M は homology lens space であるとする．t を $H_1(M)$ の生成元、d (≥ 2) を p の約数として、ξ_d を 1 の原始 d 乗根とする．$\psi_d:\mathbf{Z}[t,t^{-1}]/(t^n-1)\to\mathbf{Q}(\xi_d)$ を $\psi_d(t)=\xi_d$ で決まる環準同形とする．任意の d に対して $\tau^{\psi_d}(M)$ の値が lens space type のとき、M そのものが *lens space type* であると定義する．Theorem 1.34 より、通常の lens space は lens space type である．

Remark 2.3. あらゆる ψ_d を取ることによって、Reidemeister torsion の上ではできる限りのことをやっているのである．なぜなら、$\mathbf{Z}[t,t^{-1}]/(t^n-1)$ からの任意の準同形 φ の像 $\mathrm{Im}\,(\varphi)$ となり得るものは、準同形定理により、$\mathbf{Z}[t,t^{-1}]/(t^n-1)$ をある ideal で割ったものである．言いかえると、$\mathbf{Z}[t,t^{-1}]$ を、(t^n-1) を含む ideal で割ったものである．これは t^n-1 の約元で生成される．さらに zero divisor を持たないよ

うにするためには、既約元で生成されるべきである．すると必然的に $\mathrm{Im}\,(\varphi) \cong \mathbf{Z}[\xi_d]$ になるのである．

Notation

$$\Delta_{r,s}(t) := \frac{(t^{rs}-1)(t-1)}{(t^r-1)(t^s-1)} \qquad ((r,s)=1),$$

ここで (r,s) は r と s の最大公約数を表す．$(r,s)=1$ は r と s が互いに素ということである．これは (r,s)-torus knot の Alexander polynomial と同じである．

Main Theorem 1 *Let $K_{r,s}$ $((r,s)=1)$ be a knot in a homology 3-sphere Σ with its Alexander polynomial $\Delta_{r,s}(t)$, and $M=\Sigma(K_{r,s};p/q)$ the result of p/q-surgery along $K_{r,s}$ where $|p|,|r|,|s|\geq 2$ and $q\neq 0$. Then M is of lens space type if and only if the following (1) and (2) hold.*

(1) $(p,r)=1$, *and* $(p,s)=1$, *and*

(2) $r\equiv \pm 1 \pmod p$ *or* $s\equiv \pm 1 \pmod p$ *or* $qrs \equiv \pm 1 \pmod p$.

この証明のための準備をする．

Theorem 2.4. (Franz [6]) *Let ζ be a primitive n-th root of unity, S the set of non-zero divisors in $\mathbf{Z}/n\mathbf{Z}$. Let $\{a_j\ (j\in S)\}$ be integers satisfying the following conditions:*

(1) $a_{-j}=a_j$, (2) $\displaystyle\sum_{j\in S} a_j = 0$, (3) $\displaystyle\prod_{i\in S}(\zeta^i-1)^{a_i}=1$.

Then $a_j=0$ for all $j\in S$.

この定理の証明は L 関数の理論を用いる．

Definition 2.5. (Norm of an algebraic number)

$F/\mathbf{Q}$ を有限次 Galois 拡大とする．$\alpha\in F$ のこの拡大に関する *norm* $N_{F/\mathbf{Q}}(\alpha)$ は以下で定義される．

$$N_{F/\mathbf{Q}}(\alpha) := \prod_{\sigma\in \mathrm{Gal}\,(F/\mathbf{Q})} \sigma(\alpha).$$

要するに Galois 群でクルクル回してザッと掛けたものである．同じ α の norm でも拡大が変われば値も変わることに注意. 例えば、

$$N_{\mathbf{Q}(\sqrt{2})/\mathbf{Q}}(\sqrt{2}) = 2, \quad N_{\mathbf{Q}(\sqrt{2}+i)/\mathbf{Q}}(\sqrt{2}) = 4$$

である．しかし、α の最小多項式の定数項のベキの ± 1 倍であることには変わりがない．この他、norm に関して成り立つ基本的なことをまとめておく．

Notation 拡大 $F/\mathbf{Q}$ が明らかなときは、$N_{F/\mathbf{Q}}(\alpha)$ を単に $N(\alpha)$ と表す. 以降の話では F は円分体（cyclotomic field）であることが主である. 円分体とは $\mathbf{Q}(\zeta)$ (ζ は 1 のベキ根) のことである.

Proposition 2.6. *Let $F/\mathbf{Q}$ be a finite Galois extension, and α an element of F. We denote $N_{F/\mathbf{Q}}(\alpha)$ by $N(\alpha)$. Then*

(1) $N(\alpha) \in \mathbf{Q}$,

(2) *if α is an algebraic integer (i.e. its minimal polynomial is a monic polynomial over $\mathbf{Z}$), then $N(\alpha) \in \mathbf{Z}$,*

(3) *if α is an algebraic integer, then α^{-1} is also an algebraic integer if and only if $N(\alpha) = \pm 1$.*

Proof of Main Theorem 1

Case 1. $(p,r)=1$ かつ $(p,s)=1$ のとき

まず条件について確認しておく. 以下の ①, ② は同値である.

① $(p,r)=1$ かつ $(p,s)=1$,

② 任意の p の約数 d (≥ 2) に対して $(d,r)=1$ かつ $(d,s)=1$.

d (≥ 2) を p の約数として、ξ を 1 の原始 d 乗根とする.

$\tau^{\psi_d}(M) = \Delta_{r,s}(\xi)(\xi-1)^{-1}(\xi^{\bar{q}}-1)^{-1} = (\xi^{rs}-1)(\xi^r-1)^{-1}(\xi^s-1)^{-1}(\xi^{\bar{q}}-1)^{-1}.$

この値が $\pm\xi^m(\xi^i-1)^{-1}(\xi^j-1)^{-1}$ $((i,d)=1,\ (j,d)=1)$ となる必要十分条件を求める. つまり、

$$(\xi^{rs}-1)(\xi^i-1)(\xi^j-1) = \pm\xi^m(\xi^r-1)(\xi^s-1)(\xi^{\bar{q}}-1)$$

となる必要十分条件を求める. 両辺に複素共役をかけて整理すると、

$$\begin{aligned}&(\xi^{rs}-1)(\xi^{-rs}-1)(\xi^i-1)(\xi^{-i}-1)(\xi^j-1)(\xi^{-j}-1)\\ &=(\xi^r-1)(\xi^{-r}-1)(\xi^s-1)(\xi^{-s}-1)(\xi^{\bar{q}}-1)(\xi^{-\bar{q}}-1).\end{aligned}$$

Theorem 2.4 より、

$$\{rs,-rs,i,-i,j,-j \pmod d\} = \{r,-r,s,-s,\bar{q},-\bar{q} \pmod d\}.$$

(i) $rs \equiv \pm s \pmod d$ または $rs \equiv \pm r \pmod d$ のとき

$r \equiv \pm 1 \pmod d$ または $s \equiv \pm 1 \pmod d$ と同値である.

(ii) $rs \equiv \pm\bar{q} \pmod d$ のとき

$qrs \equiv \pm 1 \pmod{d}$ と同値である.

$d = p$ のときに、$r \equiv \pm 1 \pmod{d}$ または $s \equiv \pm 1 \pmod{d}$ または $qrs \equiv \pm 1 \pmod{d}$ であれば、他の d (≥ 2) でも成り立つ.

ちなみに、(i) のとき、$\tau^{\psi_p}(M) = \tau^{\psi_p}(L(p,q))$, (ii) のとき、$\tau^{\psi_p}(M) = \tau^{\psi_p}(L(p, r\bar{s}))$.

Case 2. $(p, r) = d \geq 2$ のとき

この条件が成り立てば、$(d, s) = 1$ であることを確認しておく.

$p = p'd, r = r'd$ とする. t を不定元とする.

$$\Delta_{r,s}(t) = \frac{(t^{rs}-1)(t-1)}{(t^r-1)(t^s-1)} = \frac{\dfrac{t^{rs}-1}{t^d-1}\cdot(t-1)}{\dfrac{t^r-1}{t^d-1}\cdot(t^s-1)}$$

$$= \frac{t^{(p's-1)d}+\cdots+t^d+1}{t^{(r'-1)d}+\cdots+t^d+1}\cdot\frac{t-1}{t^s-1}.$$

ξ を 1 の原始 d 乗根とする. このとき、

$$\tau^{\psi_d}(M) = s(\xi^s-1)^{-1}(\xi^{\bar{q}}-1)^{-1}.$$

これが $\pm\xi^m(\xi^i-1)^{-1}(\xi^j-1)^{-1}$ $((i,d)=1,\ (j,d)=1)$ になり得ないことを示す.

円分拡大 $\mathbf{Q}(\xi)/\mathbf{Q}$ の拡大次数は $\varphi(d)$ である. （$\varphi(n)$ は Euler 関数. 3.2 を参照のこと.）$N_{\mathbf{Q}(\xi)/\mathbf{Q}}(\alpha)$ を $N(\alpha)$ と表す.

$$N(s(\xi^s-1)^{-1}(\xi^{\bar{q}}-1)^{-1}) = N(s)N(\xi^s-1)^{-1}N(\xi^{\bar{q}}-1)^{-1},$$

$$N(\pm\xi^m(\xi^i-1)^{-1}(\xi^j-1)^{-1}) = N(\pm\xi^m)N(\xi^i-1)^{-1}N(\xi^j-1)^{-1}.$$

$$N(s) = s^{\varphi(d)}\ (|s|\geq 2),\ N(\pm\xi^m) = \pm 1,$$

$$N(\xi^s-1) = N(\xi^{\bar{q}}-1) = N(\xi^i-1) = N(\xi^j-1) \neq 0$$

より、両者は等しくなり得ない. よって、$\tau^{\psi_d}(M)$ になり得ない.

以上により、Case 2 は除かれて、Case 1 のみが残る. □

3. Rational surgery along a genus 1 knot

3.1. Goda-Teragaito's Theorem

Main Theorem 2 の動機となった定理を述べる. S^3 内の hyperbolic knot に沿う surgery を調べた結果の一部で、genus 1 knot から lens space を生じたならば、その knot は trefoil であるであることを述べたものである.

Theorem 3.1. (Goda-Teragaito [7]) *Let K be a genus 1 knot in S^3. If a rational surgery along K yields a lens space, then K is the trefoil.*

3.2. Cyclotomic polynomial

Main Theorem 2 の証明には、円分多項式にまつわる事実を使う．証明は [29] などを参照されたい．実は 4 章（今回は割愛する）で利用する結果も含んでいる．念のために記しておく．

Definition 3.2. (Euler function)

$\varphi : \mathbf{N} \to \mathbf{N}$ を $\varphi(n) := \left|(\mathbf{Z}/n\mathbf{Z})^\times\right| \ (n \geq 2),\ \varphi(1) = 1$

で定義したものを *Euler function* という．

Proposition 3.3. (1) *If* $(m, n) = 1$*, then* $\varphi(mn) = \varphi(m)\varphi(n)$*.*

(2) *Let* p *be a prime number, and* $r \geq 1$*.* $\varphi(p^r) = p^r - p^{r-1} = p^{r-1}(p-1)$*.*

(3) *Let* $n = \prod_{i=1}^{m} p_i^{r_i}$ *be a prime factorization of* n*.* $\varphi(n) = \prod_{i=1}^{m} p_i^{r_i - 1}(p_i - 1)$*.*

(4) *If* $n \geq 3$*, then* $\varphi(n)$ *is even.*

(5) $n = \sum_{d|n} \varphi(d) = \sum_{d|n} \varphi\left(\frac{n}{d}\right)$*.*

Definition 3.4. (Möbius function)

$$\mu : \mathbf{N} \to \{-1, 0, 1\} \text{ を } \mu(n) := \begin{cases} 1 & (n = 1) \\ (-1)^m & (n = p_1 \cdots p_m; \text{互いに異なる素数の積}) \\ 0 & (\text{その他}) \end{cases}$$

で定義したものを *Möbius function* という．

Proposition 3.5. (1) *If* $(m, n) = 1$*, then* $\mu(mn) = \mu(m)\mu(n)$*.*

(2) $\varphi(n) = \sum_{d|n} \frac{n}{d} \cdot \mu(d) = \sum_{d|n} d \cdot \mu\left(\frac{n}{d}\right)$*.*

(3) *Let* ζ *be a primitive* n*-th root of unity.* $\sum_{i \in (\mathbf{Z}/n\mathbf{Z})^\times} \zeta^i = \mu(n)$*.*

Definition 3.6. (Cyclotomic polynomial)

ζ を 1 の原始 n 乗根とするとき、

$$\mathbf{\Phi}_n(x) := \prod_{i \in (\mathbf{Z}/n\mathbf{Z})^\times} (x - \zeta^i)$$

を *n-th cyclotomic polynomial* と定義する．

Proposition 3.7. (1) $\mathbf{\Phi}_1(x) = x - 1$*.*

(2) $\mathbf{\Phi}_n(x)$ *is an irreducible monic polynomial over* $\mathbf{Z}$*.*

(3) *The degree of* $\mathbf{\Phi}_n(x)$ *is* $\varphi(n)$.

(4) $x^{\varphi(n)}\mathbf{\Phi}_n(x^{-1}) = \mathbf{\Phi}_n(x) \quad (n \geq 2)$.

(5) *Let* p *be a prime number, and* $r \geq 1$.

$$\mathbf{\Phi}_{p^r}(x) = x^{p^{r-1}(p-1)} + x^{p^{r-1}(p-2)} + \cdots + x^{p^{r-1}} + 1.$$

(6) $x^n - 1 = \prod_{d|n} \mathbf{\Phi}_d(x)$.

(7) $\mathbf{\Phi}_n(1) = \begin{cases} 0 & (n = 1) \\ p & (n = p^r; p \text{ is a prime number}) \\ 1 & (\text{otherwise}) \end{cases}$

(8) $\mathbf{\Phi}_n(x) = \prod_{d|n} (x^{\frac{n}{d}} - 1)^{\mu(d)}$.

(9) *Let* $n = \prod_{i=1}^{m} p_i^{r_i}$ *be a prime factorization of* n.

$$\mathbf{\Phi}_n(x) = \mathbf{\Phi}_{p_1 \cdots p_m}(x^{p_1^{r_1-1} \cdots p_m^{r_m-1}})$$

.

3.3. Main Theorem 2

次数 2 の Alexander polynomial を持つ knot に沿う surgery がいつ lens space type になり得ないかを述べたのが Main Theorem 2 である.

Notation $\Delta_n(t) := n(t-1)^2 + t = nt^2 - (2n-1)t + n \quad (n \neq 0)$.

Main Theorem 2 *Let* K *be a knot in a homology* 3*-sphere* Σ *with its Alexander polynomial* $\Delta_K(t) = \Delta_n(t)$, *and* $M = \Sigma(K; p/q)$ *the result of* p/q*-surgery along* K *where* $|p| \geq 2$ *and* $q \neq 0$. *Let* d (≥ 2) *be a divisor of* p, ξ_d *a primitive* d*-th root of unity,* $\psi_d : \mathbf{Z}[t, t^{-1}]/(t^p - 1) \to \mathbf{Q}(\xi_d)$ *a homomorphism such that* $\psi_d(t) = \xi_d$, *and* $\tau^{\psi_d}(M)$ *the Reidemeister torsion associated to* ψ_d. *Then the following* (1) *and* (2) *hold.*

(1) *If* $n \leq -1$, *then* $\tau^{\psi_p}(M)$ *is not of lens space type.*

(2) *If* $|n| \geq 2$ *and* d *is a prime number, then* $\tau^{\psi_d}(M)$ *is not of lens space type.*

これにより以下が導かれる.

Corollary 3.8. *In the same assumption as Main Theorem 2, if* M *is of lens space type, then*

$$\Delta_K(t) = t^2 - t + 1 \quad (n = 1).$$

この結果は Theorem 3.1 の代数的な翻訳になっていて、さらに以下の Ozsváth-Szabó [17] の結果の一部の拡張になっている.

Theorem 3.9. (Ozsváth-Szabó [17]) *Let K be a knot in S^3, and $M = S^3(K;p)$ the result of p-surgery along K where p is an integer. If M is a lens space, then the Alexander polynomial of K is the following form*

$$\Delta_K(t) = (-1)^m + \sum_{j=1}^{m}(-1)^{m-j}(t^{s_j} + t^{-s_j}),$$

where $0 < s_1 < s_2 < \cdots < s_m$.

つまり、次数 2 の場合に限ると、S^3 を homology 3-sphere Σ に、整数 surgery を有理数 surgery に拡張している.

Main Theorem 1 の Case 2 の証明を観察すると、$\tau^{\psi_d}(M)$ の norm が lens space type な数の norm と一致するための条件がわかる.

Lemma 3.10. *Let K be a knot in a homology 3-sphere Σ with its Alexander polynomial $\Delta_K(t)$, and $M = \Sigma(K;p/q)$ the result of p/q-surgery along K where $|p| \geq 2$ and $q \neq 0$. If M is of lens space type, then*

$$N_{\mathbf{Q}(\xi_d)/\mathrm{Q}}(\Delta_K(\xi_d)) = \pm 1 \quad (d|p,\ d \geq 2).$$

$\Delta_K(\xi_d)$ が代数的整数環 $\mathbf{Z}[\xi_d]$ の中で unit (Proposition 2.6 (3)) であることを言っている. 実は $\Delta_K(t)$ が $\mathbf{Z}[\mathbf{Z}_n]$ の中で unit であることまで言える.

Definition 3.11. (Norm polynomial)

ζ を 1 の原始 p 乗根とするとき、

$$f_p(n) := N_{\mathbf{Q}(\zeta)/\mathrm{Q}}(\Delta_n(\zeta))$$

を *norm polynomial* と定義する. これは n についての $\mathbf{Z}$ 係数の多項式となる.

$\Delta_n(t) = 0$ の根を α_1, α_2 とする. このとき以下の変形が重要である.

$$f_p(n) = \prod_{i \in (\mathbf{Z}/n\mathbf{Z})^\times} n(\zeta^i - \alpha_1)(\zeta^i - \alpha_2) = n^{\varphi(p)}\mathbf{\Phi}_p(\alpha_1)\mathbf{\Phi}_p(\alpha_2).$$

Main Theorem 2 よりも強い主張である以下を示す.

Theorem 3.12. (1) *If $n \leq -1$, then $f_p(n) \neq \pm 1$.*

(2) *If $|n| \geq 2$ and p is a prime number, then $f_p(n) \neq \pm 1$.*

特に $p=2$ のとき、$f_2(n)=\Delta_n(-1)=4n-1$ より、$n\neq 0$ で $f_2(n)\neq \pm 1$ となる．以下、$p\geq 3$ を仮定する．

Proposition 3.13. *The degree of* $f_p(n)$ *is* $\varphi(p)$.

Proof $\Delta_n(\zeta^i)=(1-\zeta^i)^2 n+\zeta^i \quad (i\in(\mathbf{Z}/n\mathbf{Z})^\times)$ は n の 1 次式である． □

Lemma 3.14. *If* $p\geq 3$, *then there exists a polynomial* $g_p(n)$ *of* n *over* $\mathbf{Z}$ *such that* $f_p(n)=\{g_p(n)\}^2$.

Proof $\Delta_n(\zeta)=\zeta^2\Delta_n(\zeta^{-1})$.

$$\delta(\zeta):=\frac{\Delta_n(\zeta)}{\zeta}=\frac{\Delta_n(\zeta^{-1})}{\zeta^{-1}}$$

とおく．このとき、$\delta(\zeta)\in\mathbf{Q}(\zeta+\zeta^{-1})$ である．

$\zeta\neq\zeta^{-1}$ より、$[\mathbf{Q}(\zeta):\mathbf{Q}(\zeta+\zeta^{-1})]=2$ なので、$[\mathbf{Q}(\zeta+\zeta^{-1}):\mathbf{Q}]=\varphi(p)/2$.

$$g_p(n):=N_{\mathbf{Q}(\zeta+\zeta^{-1})/\mathbf{Q}}(\delta(\zeta))$$

とすればよい． □

$f_p(n)$, $g_p(n)$ を以下のように展開しておく．

$$f_p(n)=\sum_{i=0}^{\varphi(p)} a_i n^i, \qquad g_p(n)=\sum_{j=0}^{\varphi(p)/2} b_j n^j.$$

Lemma 3.15. $a_{\varphi(p)}=\{\mathbf{\Phi}_p(1)\}^2$ *and* $a_0=1$.

Proof $a_{\varphi(p)}=N_{\mathbf{Q}(\zeta)/\mathbf{Q}}((1-\zeta)^2)=\{\mathbf{\Phi}_p(1)\}^2,\ a_0=N_{\mathbf{Q}(\zeta)/\mathbf{Q}}(\zeta)=1.$ □

Lemma 3.16. $f_p(n)$ *and* $g_p(n)$ *are alternating polynomials.*

Proof

$$\delta(\zeta)=\frac{\Delta_n(\zeta)}{\zeta}=1-2\{1-\cos(2\pi k/p)\}n.$$

ここで、$(k,p)=1$ なので、これらの積は alternating. □

Proof of Theorem 3.12. (1) $n\leq -1$ のとき、$f_p(n)\geq a_{\varphi(p)}+a_0\geq 1+1=2$ より、$f_p(n)\neq\pm 1$. □

Corollary 3.17. *We can take* $b_{\varphi(p)/2}=\mathbf{\Phi}_p(1)$ *and* $b_0=(-1)^{\varphi(p)/2}$.

Proof Lemma 3.15, Lemma 3.16 より得られる． □

Lemma 3.18. $a_1=2\mu(p)-2\varphi(p)$. *In particular, if* p *is an odd prime number, then* $a_1=-2p$ *and* $b_1=(-1)^{\varphi(p)/2-1}p$.

Proof $a_1=\displaystyle\sum_{i\in(\mathbf{Z}/p\mathbf{Z})^\times}(1-\zeta^i)^2\frac{N_{\mathbf{Q}(\zeta)/\mathbf{Q}}(\zeta)}{\zeta^i}=\sum_{i\in(\mathbf{Z}/p\mathbf{Z})^\times}(\zeta^i+\zeta^{-i}-2)$

$$= 2 \sum_{i \in (\mathbf{Z}/p\mathbf{Z})^\times} \zeta^i - 2\varphi(p).$$

Proposition 3.5. (3) より、$a_1 = 2\mu(p) - 2\varphi(p)$.

特に p が奇素数のとき、$a_1 = 2 - 2(p-1) = -2p$.

$$(\cdots + b_1 n + b_0)^2 = \cdots + 2b_0 b_1 n + b_0^2$$

より、$b_1 = (-1)^{\varphi(p)/2-1} p$. □

Lemma 3.19. *If p is a prime number, then $f_p(n) = n^p(\alpha_1^p - 1)(\alpha_2^p - 1)$.*

Proof $\dfrac{n^p(\alpha_1^p - 1)(\alpha_2^p - 1)}{f_p(n)} = n(\alpha_1 - 1)(\alpha_2 - 1) = \Delta_n(1) = 1.$ □

Lemma 3.20. *If p is an odd prime number, then*

$$b_j \equiv 0 \pmod p \quad (j = 1, \cdots, \varphi(p)/2).$$

Proof $f_p(n) = n^p(\alpha_1^p - 1)(\alpha_2^p - 1) \equiv n^p(\alpha_1 - 1)^p(\alpha_2 - 1)^p$
$= \{n(\alpha_1 - 1)(\alpha_2 - 1)\}^p = 1^p = 1 \ (\mathrm{mod}\,(p))$.

ここで (p) は、p で生成される n の多項式環 $\mathbf{Z}[n]$ の ideal.

$\mathbf{Z}[n]/(p) \cong (\mathbf{Z}/p\mathbf{Z})\,[n]$ は UFD なので、$f_p(n) = \{g_p(n)\}^2 \equiv 1 \ (\mathrm{mod}\ (p))$ から $g_p(n) \equiv \pm 1 \ (\mathrm{mod}\,(p))$ となる. これは、

$$b_j \equiv 0 \pmod p \quad (j = 1, \cdots, \varphi(p)/2)$$

を意味する. □

Proof of Theorem 3.12. (2) $p = 2$ の場合は終わっているので、p は奇素数とする. Lemma 3.20 より、$\mathbf{Z}$ 上の多項式 $h_p(n)$ が存在して、

$$g_p(n) = pn \cdot h_p(n) + b_0.$$

$p \geq 3$ より、$f_p(n) = \pm 1$ と $h_p(n) = 0$ は同値.

$$h_p(n) = \sum_{k=0}^{N} c_k n^k$$

とするとき、Corollary 3.17, Lemma 3.18 より、$c_N = 1, c_0 = \pm 1$. これより、$h_p(n) = 0$ ならば、$n = \pm 1$. （高校数学！）よって $|n| \geq 2$ のとき、$h_p(n) \neq 0$. つまり、$f_p(n) \neq \pm 1$. □

例えば、$h_3(n) = 1,\ h_5(n) = n - 1,\ h_7(n) = (n-1)^2,\ h_{11}(n) = (n-1)(n^3 - 4n^2 + 3n - 1)$.

Corollary 3.21. *Let K be a knot in a homology 3-sphere Σ with its Alexander polynomial $\Delta_K(t)$, and $M = \Sigma(K; p/q)$ the result of p/q-surgery along K where*

$|p| \geq 2$ and $q \neq 0$. If $\Delta_K(t)$ can be divided by $\Delta_n(t)$ $(n \neq 0, 1)$, then M is not of lens space type.

Proof $N(\Delta_K(\zeta))$ は $N(\Delta_n(\zeta))$ で割り切れて、$N(\Delta_n(\zeta)) \neq \pm 1$ である. □

Question 3.22. *If $\Sigma(K; p/q)$ is of lens space type, then is $\Delta_K(t)$ a product of cyclotomic polynomials ?*

せっかく Question にしておきながら恐縮だが、これには反例がある. K を $(-2, 3, 7)$-pretzel knot とする (Figure 10). Fintushel-Stern[4] により、K に沿う 18-, 19-surgery が lens space になることが知られている. ところが、

$$\Delta_K(t) = t^{10} - t^9 + t^7 - t^6 + t^5 - t^4 + t^3 - t + 1$$

は **Z** 上 irreducible だが、1 のベキ根を解に持たないので反例である (cf. [9]).

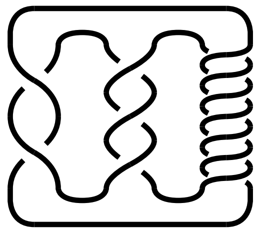

FIGURE 10. $(-2, 3, 7)$-pretzel knot

4. Generalizations and Applications

この章は割愛させていただきます. すいません.

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Pseudo-Anosov braids on the 2-sphere

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Abstract. A correspondence between braids on the 2-disk and those on the 2-sphere is naturally induced from the inclusion map of the 2-disk into the 2-sphere. A natural necessary condition for a pseudo-Anosov braid on the 2-disk so that the corresponding braid on 2-sphere is also pseudo-Anosov. It is shown that this condition is not sufficient in general.

1. Preliminary

The main subject of this talk is the natural correspondence between braids on the 2-disk and those on the 2-sphere. Let b be an n-braid on $\mathbf{D^2}$ with $n \geq 3$. A correspondence between braids on $\mathbf{D^2}$ and those on $\mathbf{S^2}$ is naturally induced from the inclusion map of $\mathbf{D^2}$ into $\mathbf{S^2}$. We denote by $\hat{b}$ the braid on $\mathbf{S^2}$ corresponding to b. Then it is natural to ask:

Question 1. *What happens under this correspondence?*

For example it has been determined which braids on $\mathbf{D^2}$ become trivial under the correspondence. Please refer to [1] and [8] for the theory of braids.

In this talk, we concentrate our focus on the behavior of the *Nielsen-Thurston type* under the correspondence.

1.1. **Nielsen-Thurston type.** In the following, we give a rough explanation on the Nielsen-Thurston types of braids.

First recall that the definition for surface automorphisms. Let $\Sigma_{g,p,b}$ denote a compact orientable surface of genus g with p distinguished points and b boundary components.

Definition 1 ([11]; [3], [2])**.** An orientation preserving homeomorphism of $\Sigma_{g,p,b}$ is;

(1) *periodic* if whose some power is equal to the identity,
(2) *reducible* if it leaves an essential 1-submanifold of $\Sigma_{g,p,b}$ invariant (a 1-submanifold of $\Sigma_{g,p,b}$ is called *essential* if each component is homotopically

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non-trivial and not boundary-parallel, and no two components are homotopic),

(3) *pseudo-Anosov* if for the map f, there exist a pair of transverse measured foliations $(\mathcal{F}^s, \mu^s)$, $(\mathcal{F}^u, \mu^u)$ such that $f(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda\mu^s)$ and $f(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda^{-1}\mu^u)$ for some $\lambda > 1$.

Mainly due to Nielsen and Thurston, the following trichotomy has been established.

Fact 1 (Nielsen-Thurston classification ([11]. see also [3] or [2].))**.** *Suppose that* $2g-2+p+b > 0$ *holds. Then any orientation preserving homeomorphism of* $\Sigma_{g,p,b}$ *is isotopic to a periodic map, a reducible map, or a pseudo-Anosove map.*

Remark that this trichotomy is invariant under conjugation.

Now we give a definition of the Nielsen-Thurston types of braids.

Definition 2. Let b be an n-braid either on the 2-disk $\mathbf{D^2}$ or on the 2-sphere $\mathbf{S^2}$ with $n \geq 3$.

- There exists a horizontal-level preserving homeomorphism Φ of $\mathbf{D^2} \times [0,1]$ or $\mathbf{S^2} \times [0,1]$ such that $\Phi(x,1) = (x,1)$ and $\Phi(b)$ become the trivial braid. Then $\Phi|_{\mathbf{D^2}\times\{0\}}$ or $\Phi|_{\mathbf{S^2}\times\{0\}}$ yields a homeomorphism of $\mathbf{D^2}$ or $\mathbf{S^2}$, which is determined up to isotopy. We call this homeomorphism *the homeomorphism associated to* b and denote it by f_b.
- The braid b is *periodic*, *reducible*, or *pseudo-Anosov* if f_b is isotopic to a periodic map, a reducible map, or a pseudo-Anosov map, respectively.

It can be easily seen that

- if a braid b on $\mathbf{D^2}$ is periodic, then the corresponding braid $\hat{b}$ on $\mathbf{S^2}$ is periodic,
- if a braid b on $\mathbf{D^2}$ is reducible, then the corresponding braid $\hat{b}$ on $\mathbf{S^2}$ is reducible or $\hat{b}$ is equivalent to a conjugate of a braid which has one isolated string.

Therefore the following question arise.

Question 2. *For which pseudo-Anosov braid* b *on* $\mathbf{D^2}$*, is* (*not*) *the corresponding braid* $\hat{b}$ *on* $\mathbf{S^2}$ *pseudo-Anosov?*

1.2. **Problem.** Concerning this question, the next observation was given by J. Los (in private communication).

Fact 2 ([7]). *Let b be a pseudo-Anosov braid on $\mathbf{D^2}$. If the corresponding braid $\hat{b}$ on $\mathbf{S^2}$ is NOT pseudo-Anosov, then the invariant measured foliation of f_b has a 1-prong singularity on the boundary $\partial\mathbf{D^2}$.*

For example, we have the following for 3-braids on $\mathbf{D^2}$,

Fact 3 ([9], see also [10]). *Let b be a 3-braid on $\mathbf{D^2}$.*

(1) The corresponding braid $\hat{b}$ of $\mathbf{S^2}$ is always periodic, in particular, is not pseudo-Anosov.
(2) The braid b is pseudo-Anosov if and only if it is conjugate to a braid $(\sigma_1\sigma_2)^{2k}P(\sigma_1^{-1},\sigma_2)$ for some integer k and a positive word P.
(3) When b is pseudo-Anosov, then the invariant measured foliation of f_b has a 1-prong singularity on the boundary $\partial\mathbf{D^2}$.

In the fact above, $\sigma_1,\ldots,\sigma_{n-1}$ denote the standard Artin generators of n-braids on $\mathbf{D^2}$.

Thus the next problem can be considered.

Question 3. *Is the converse of Fact 2 true? That is, for a pseudo-Anosov braid b on $\mathbf{D^2}$, if the invariant measured foliation of f_b has a 1-prong singularity on the boundary $\partial\mathbf{D^2}$, then is the corresponding braid $\hat{b}$ on $\mathbf{S^2}$ not pseudo-Anosov?*

2. Results

Our main result is as follows: Question 3 is negatively answered.

Theorem. *Let $b_{n,k}$ be the n-braid on $\mathbf{D^2}$ given by*

$$\sigma_1\sigma_2\cdots\sigma_k(\sigma_{k+1})^{-1}\cdots(\sigma_{n-1})^{-1}$$

where $n\geq 4$ and $1\leq k\leq n-2$.

(1) The braid $b_{n,k}$ is pseudo-Anosov for all $n\geq 4$ and $1\leq k\leq n-2$.
(2) The invariant measured foliation of $f_{b_{n,k}}$ has a 1-prong singularity on the boundary $\partial\mathbf{D^2}$.
(3) The braid $\widehat{b_{n,k}}$ on $\mathbf{S^2}$ is periodic if and only if n is odd and $k=(n-1)/2$.
(4) The braid $\widehat{b_{n,k}}$ on $\mathbf{S^2}$ is reducible if and only if n is even and $k=(n-2)/2,n/2$.

Corollary. *Let $b_{n,k}$ be the n-braid on $\mathbf{D^2}$ as in Theorem 2. Then the corresponding braid $\widehat{b_{n,k}}$ on $\mathbf{S^2}$ is not pseudo-Anosov if and only if $k=(n-2)/2,(n-1)/2,n/2$.*

We end this note by giving some keys to our proof.

(1) and (2) These are achieved by actual constructions of invariant transverse measured foliations for $f_{b_{n,k}}$. Essentially this was done in [5].

the 'if' part of (3) and (4) One can check these part by 'hand'; by drawing and manipulating figures.

the 'only if' part of (3) The fact we use here is: if $\widehat{b_{n,k}}$ on $\mathbf{S^2}$ is periodic, then it is conjugate to the braid presented by

$$({\sigma_1}^{\varepsilon}{\sigma_2}^{\varepsilon}\cdots{\sigma_{n-2}}^{\varepsilon}{\sigma_{n-1}}^{\varepsilon})^m$$

where $\varepsilon = 1$ or -1, and $m \in \mathbb{Z}$. Then the assertion follows from [8, Chap. 11, Proposition 2.3].

the 'only if' part of (4) By identifying $\mathbf{S^2} \times \{0\}$ and $\mathbf{S^2} \times \{1\}$ of $\mathbf{S^2} \times [0,1]$, we obtain a knot $\widehat{K_{n,k}}$ from $\widehat{b_{n,k}}$ in $\mathbf{S^2} \times S^1$. Note that if $\widehat{b_{n,k}}$ on $\mathbf{S^2}$ is reducible, then the complement $\mathbf{S^2} \times [0,1] \setminus \widehat{K_{n,k}}$ contains an essential torus.

Let $L_{n,k}$ be the 2-component link in the 3-sphere represented as the closure of the $(n+1)$-braid

$$\sigma_1\sigma_2\cdots\sigma_{n-1}\sigma_n\cdot\sigma_1\sigma_2\cdots\sigma_k(\sigma_{k+1})^{-1}\cdots(\sigma_{n-1})^{-1}\cdot\sigma_n\sigma_{n-1}\cdots\sigma_2\sigma_1.$$

Then it is easily seen that the complement $\mathbf{S^2} \times [0,1] \setminus \widehat{K_{n,k}}$ is homeomorphic to the 3-manifold which is obtained by 0-Dehn filling on one component of $L_{n,k}$. Thus it suffices to study the toroidal Dehn surgeries on one component of $L_{n,k}$. Moreover we note that the link $L_{n,k}$ is a two-bridge link: It has the Conway form $C(2n-2k-1, 2k+1)$. Then, based on the work [4], we can show that 0-Dehn filling on one component of $L_{n,k}$ is toroidal if and only if n is even and $k = (n-2)/2, n/2$. We remark that similar results were obtained in [6] independently.

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On the A-polynomial of a knot

Naoko Tamura

1.1. Definition of the A-polynomial

Cooper-Culler-Gillet-Long-Shalen

"Plane curves associated to character varieties of 3-manifolds."

Invent. Math ('94)

Definition Let

M : *compact* 3-mfd *s.t.* $\partial M = T^2$,

$\pi_1(\partial M) \ni \lambda,\ \mu$

$R := Hom(\pi_1(M),\ SL(2, \mathbf{C}))$

$\cup$

$$R_U := \left\{ \rho \in R \,|\, \rho(\lambda) = \begin{pmatrix} l & * \\ 0 & l^{-1} \end{pmatrix}, \rho(\mu) = \begin{pmatrix} m & * \\ 0 & m^{-1} \end{pmatrix} \right\}$$

$$Z[l, m] \ni {}^{\exists} f(l, m) = 0,$$

then the A-polynomial of M is defined as

$$A(l, m) = \frac{f(l, m)}{(l - 1)}.$$

K : *a knot*, $M = S^3 - N(K)$: *a knot complement*

$$\Rightarrow A_K(l, m) : A\text{-}polynomial\ of\ K$$

1.2. Computation of the A-polynomial

For example

$K = 3_1$

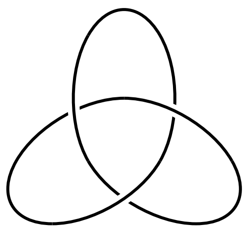

FIGURE 1

$$\begin{cases} \pi_1(K) = < \mu, x \mid \mu x \mu = x \mu x > \\ \mu^4 \lambda = \mu x \cdot x \mu \end{cases}$$

After conjugation and after replacing x by x^{-1} if nenessary we can assume that

$$\rho(x) = \begin{pmatrix} m & 0 \\ t & m^{-1} \end{pmatrix}.$$

Then

$$\begin{cases} \rho(\mu)\rho(x)\rho(\mu) = \rho(x)\rho(\mu)\rho(x) \\ \rho(\mu)^4\rho(\lambda) = \rho(\mu)\rho(x)\rho(x)\rho(\mu) \end{cases}$$

$$\Leftrightarrow \begin{cases} 1 - m^2 + m^4 + m^2 t = 0 \\ -m^4 + lm^4 - t - m^2 t = 0, \end{cases}$$

from these equation we obtain the A-polynomial of K.

$$A_K(l, m) = 1 + lm^6$$

$$A_{4_1} = -m^4 + l(m^8 - m^6 - 2m^4 - m^2 + 1) - l^2 m^4$$

$$
\begin{aligned}
A_{5_2} &= -l^3 + (1 - 2m^2 - 2m^4 + m^8 - m^{10})l^2 \\
&\quad +(-m^4 + m^6 - 2m^{10} - 2m^{12} + m^{14})l - m^{14} \\
A_{6_1} &= -m^8 + (1 - m^2 - 3m^8 - 3m^{10} + 2m^{12})l \\
&\quad +(-1 + 3m^2 + m^4 - 3m^6 - 6m^8 \\
&\qquad -3m^{10} + m^{12} + 3m^{14} - m^{16})l^2 \\
&\quad +(2 - 3m^2 - 3m^4 - m^{10} + m^{12})m^4 l^3 - m^8 l^4 \\
A_{7_2} &= m^{22} + (1 - m^2 + 3m^{10} + 4m^{12} - 2m^{14})m^8 l \\
&\quad +(-2 + 5m^2 + m^4 - 4m^6 + 6m^{10} \\
&\qquad +5m^{12} + 2m^{14} - 4m^{16} + m^{18})m^4 l^2 \\
&\quad +(1 - 4m^2 + 2m^4 + 5m^6 + 6m^8 \\
&\qquad -4m^{12} + m^{14} + 5m^{16} - 2m^{18})l^3 \\
&\quad +(-2 + 4m^2 + 3m^4 - m^{12} + m^{14})l^4 + l^5
\end{aligned}
$$

$$
\begin{aligned}
A_{8_2} = \; & m^{72} + (-5 + 9m^2 + 7m^4 - 3m^6 - m^{12} + 2m^{14} - m^{16})lm^{60} \\
& +(10 - 32m^2 - m^4 + 56m^6 + 17m^8 - 28m^{10} - m^{12} + 14m^{14} - 4m^{16} \\
& -8m^{18} + 7m^{20} - 2m^{22})l^2 m^{48} \\
& +(-10 + 42m^2 - 24m^4 - 87m^6 + 29m^8 + 143m^{10} + 33m^{12} - 77m^{14} \\
& -17m^{16} + 29m^{18} - 2m^{20} + m^{22} - 8m^{24} + 5m^{26} - m^{28})l^3 m^{36} \\
& +(5 - 24m^2 + 26m^4 + 36m^6 - 43m^8 - 108m^{10} + 47m^{12} + 192m^{14} \\
& +47m^{16} - 108m^{18} - 43m^{20} + 36m^{22} + 26m^{24} - 24m^{26} + 5m^{28})l^4 m^{24} \\
& +(-1 + 5m^2 - 8m^4 + m^6 - 2m^8 + 29m^{10} - 17m^{12} - 77m^{14} + 33m^{16} \\
& +143m^{18} + 29m^{20} - 87m^{22} - 24m^{24} + 42m^{26} - 10m^{28})l^5 m^{12} \\
& +(-2 + 7m^2 - 8m^4 - 4m^6 + 14m^8 - m^{10} - 28m^{12} + 17m^{14} + 56m^{16} \\
& -m^{18} - 32m^{20} + 10m^{22})l^6 m^6 \\
& +(-1 + 2m^2 - m^4 - 3m^{10} + 7m^{12} + 9m^{14} - 5m^{16})l^7 + m^4 l^8
\end{aligned}
$$

2. An ideal triangulation of a knot complement

Y. Yokota

"From the Jones polynomial to the A-polynomial of hyperbolic knots."

Int. Sci. (2003)

K : 5_2, M : $S^3 - N(K)$

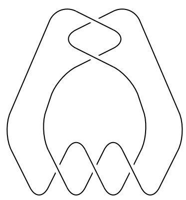

FIGURE 2

Now we construct an ideal triangulation of M, where the ideal tetrahedra is given below,

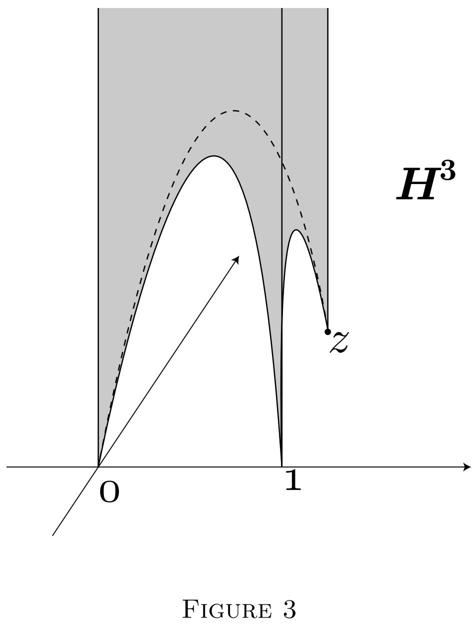

FIGURE 3

and 3 moduli of a triangle are related as follows.

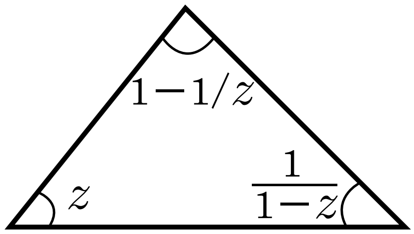

FIGURE 4

Then we have $\partial N(K)$.

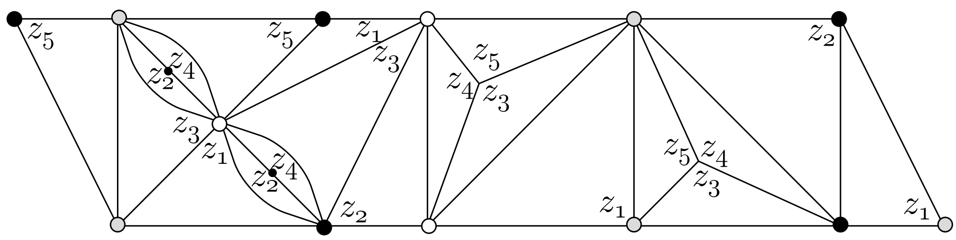

FIGURE 5

Then we have following hyperbolicity equations, where the variables l and m represent longitude and meridian.

$$\begin{aligned} m^2 &= \frac{z_1}{z_5} = \frac{(1-1/z_3)(1-z_2)}{1-z_4} = \frac{1-1/z_3}{(1-1/z_1)(1-z_5)} \\ l^2 &= (1-z_1)\cdot(1-z_4)\cdot\frac{1}{1-z_4}\cdot\frac{1}{z_3}\cdot\frac{1-z_5}{z_5}\cdot(1-z_1) \\ &\quad \times\frac{1}{1-z_3}\cdot(1-z_1)\cdot\frac{1}{z_5}\cdot\frac{1}{1-z_1}\cdot\frac{1-z_3}{z_3}\cdot\frac{1}{1-z_5} \\ &= \left\{\frac{1}{z_3 z_5}\cdot(1-z_1)\right\}^2 \end{aligned}$$

If this ideal triangulation determines a hyperbolic structure of M, the product of the moduli around each edge should be 1, that is,

$$1 = z_2 z_4 = z_3 z_4 z_5,$$

which suggest to put

$$z_1 = ym, z_2 = \frac{x}{m}, z_3 = \frac{x}{y}, z_4 = \frac{m}{x}, z_5 = \frac{y}{m},$$

Then we can rewrite the hyperbolicity equations

$$m^2 = \frac{(1 - y/x)(1 - x/m)}{1 - m/x} = \frac{1 - y/x}{(1 - y/m)(1 - 1/ym)}$$

and

$$l = \frac{m}{x} \cdot (1 - ym).$$

From these equations we have

$$\begin{aligned} A_K(l, m) = &-l^3 + (1 - 2m^2 - 2m^4 + m^8 - m^{10})l^2 \\ &+(-m^4 + m^6 - 2m^{10} - 2m^{12} + m^{14})l - m^{14}. \end{aligned}$$

3. A formula for some infinite knot families

J. Hoste, P. D. Shanahan

"A formula for the A-polynomial of twist knots"

(2002)

The twist knot is the knot shown in the following picture.

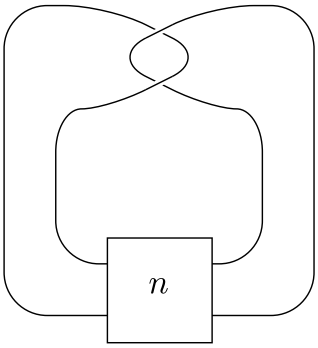

FIGURE 6

THEOREM

The A-polynomial $A_n(l, m)$ of the twist knot is given recursively by

$$A_n(l, m) = xA_{n-n/|n|}(l, m) - yA_{n-2n/|n|}(l, m),$$

where x, y and initial conditions A_{-1}, A_0, A_1, A_2 are the polynomial of l and m.

N. Tamura, Y. Yokota

"A formula for the A-polynomials of $(-2, 3, 2n+1)$-pretzel knots."

Tokyo Math. to appear.

The pretzel knot is the knot shown in the following.

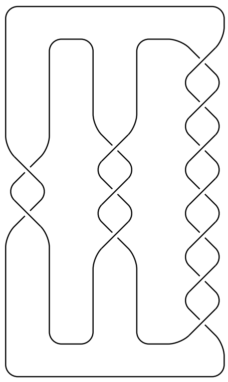

FIGURE 7

This is $(-2, 3, 7)$-pretzel knot. Then let K_n be the knot shown in the following picture.

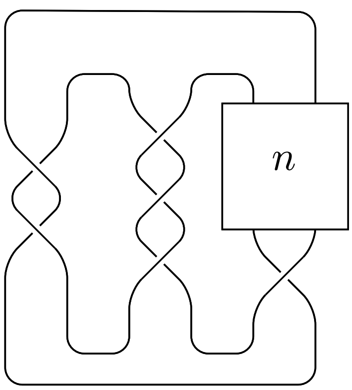

FIGURE 8

If $n = 0, 1, 2, 3$, the knot K_n are the following knots.

$K_0 = 5_1$

$$A_{K_0}(l, m) = 1 + lm^{10},$$

$K_1 = 8_{19}$

$$A_{K_1}(lm^{-4}, m) = 1 + lm^8,$$

$K_2 = 10_{124}$

$$A_{K_2}(lm^{-8}, m) = (1 + lm^7)(1 - lm^7),$$

$K_3 = (-2, 3, 7)$-pretzel knot

$$\begin{aligned} & A_{K_3}(lm^{-12}, m) \\ & \quad = 1 - (1 - 2m^2 + m^4)\, m^4\, l - (2 + m^2)\, m^{12}\, l^2 \\ & \quad +(1 + 2m^2)\, m^{24}\, l^4 + (1 - 2m^2 + m^4)\, m^{30}\, l^5 \\ & \quad -m^{38}\, l^6. \end{aligned}$$

Main Theorem

We can define $A_n(lm^{-4n}, m)$ recursively by

$$\frac{A_n}{B_n} = \frac{\gamma}{\alpha}\,\frac{A_{n-1}}{B_{n-1}} + (2 + \frac{2\gamma}{\alpha} + \frac{\beta^2}{\alpha})\,\frac{A_{n-2}}{B_{n-2}} + \frac{\gamma}{\alpha}\,\frac{A_{n-3}}{B_{n-3}} - \frac{A_{n-4}}{B_{n-4}},$$

where B_n is

$$\begin{cases} -l^2(lm^8)^{3+n}(1-m^2)^n(1+lm^6)^{3+n} & (n > 3), \\ -(lm^8)^{-(2+n)}(1-m^2)^{-(1+n)}(1+lm^6)^{2-n} & (n < 0) \end{cases}$$

and

$$\begin{aligned} \alpha &= lm^8(1-m^2)(1+lm^6), \\ \beta &= m^2 - (1-2m^2)l\,m^6 - (2+m^2)\,l^2\,m^{16} - l^3\,m^{22}, \\ \gamma &= -(1+m^4) - (2+m^2-m^4)\,l\,m^8 \\ &\quad +(-l+m^2+2m^4)\,l^2\,m^{12} + (1+m^4)\,l^3\,m^{20}. \end{aligned}$$

♠ The outline of the proof.

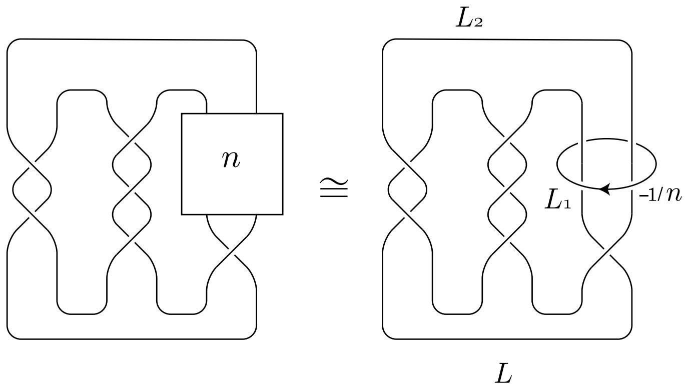

Figure 9

Legendrian curves on fiber surfaces

山本 亮介 (Ryosuke Yamamoto)
大阪大学大学院理学研究科 研究生

Abstract. I will talk a way of constructing a (positive) contact structure of closed orientable 3-manifold M from an open-book decomposition of M, and we will see a condition that a simple closed curve on a fiber surface becomes a Legendrian curve in the contact structure.

1. Preliminary

Let M be a closed smooth orientable 3-manifold, K a fibered knot in M with a fiber surface F. Let $E(K)$ denotes an exterior of K in M.

Definition 1.1. An *open-book decomposition* (F, φ) of M consists of the knot K, called the *binding*, and a fibration $\varphi : E(K) \to S^1$. A fiber surface F is called a *page*. Note that

$$E(K) \cong F \times [0,1]/(x,1) \sim (h(x),0)$$

, where $h : F \to F$ is a homeomorphism fixing ∂F pointwize.

Definition 1.2 ([1])**.** A *contact form* on M is a smooth 1-form ω such that

$$\omega \wedge d\omega \neq 0$$

at each point. A *contact structure* (M, ξ) is a 2-plane field $\xi = \ker \omega$ on M. We call a contact structure is *positive* when $\omega \wedge d\omega > 0$.

Example 1.3. Let (R^3, ξ_0) be a contact structure on R^3 defined by the contact form $\omega_0 = xdy - ydx + dz$.

$$(xdy - ydx + dz) \wedge d(xdy - ydx + dz) = 2dz \wedge dx \wedge dy > 0$$

We call this structure the *standard* contact structure on R^3 (see Figure A).

Definition 1.4. A contact structure on M is *supported* by an open-book decomposition (F, φ) if it is defined by a 1-form ω such that
(1) on each fiber F, $d\omega|_F > 0$,
(2) ω is transverse to K and orients K as the boundary of $(F, d\omega)$.

Definition 1.5. A simple closed curve γ is called *Legendrian* if for every $x \in \gamma$, $T_x\gamma \subset \xi_x$ (i.e., γ is always tangent to ξ).

2. Construction of a contact structure

The aim of this section is to construct a positive contact form ω on M supported by (F, φ) from the standard contact strucure on S^3, through a *simple cover* $p : M \to S^3$. This construction is based on [3].

- Put $D = \{(x, y) \in R^2 | x^2 + y^2 \leq 1\}$, and 1-form $\alpha_0 = xdy - ydx$ on D. Note that α_0 has the propaties follows:
  (1) $d\alpha_0 = 2dx \wedge dy$ is a volume form on D,
  (2) α_0 orients ∂D as the boundary of $(D, d\alpha_0)$.

Definition 2.1. Let S be a orientable surface with boundary and D a 2-disk. A branched covering map $p : S \to D$ is called a *simple cover* with d sheets if there is a finite set $Q \subset \operatorname{Int} D$ and each $x \in D$ has a disk neighbourhood U as follows:
(1) if $x \notin Q$ then $p|_{p^{-1}(U)}$ is a trivial d-sheeted cover,
(2) if $x \in Q$ then $p^{-1}(U)$ has $d - 1$ components, one of which is a disk projecting to U as double cover branched over x, and the others are disks projecting homeomorphically.

- Let $p : F \to D$ be a simple cover, and set $\alpha = p^*\alpha_0$. We have that (1) $d\alpha$ is a volume form on F and (2) α orients ∂F as the boundary of $(F, d\alpha)$.
- The 1-form β on $E(K) = F \times [0, 1]/(x, 1) \sim (h(x), 0)$ such that
$$\beta|_{F\times t} = (1 - t)\alpha + th^*\alpha,\ t \in [0, 1]$$
has the properties (1) and (2) in Definition 1.4, and may not be contact. Let $ds = \varphi^* d\theta$, where $d\theta$ is a volume form on S^1. For a sufficient large constant N, the form
$$\omega = \beta + Nds$$
is a contact form on $E(K)$. We can extend ω to M smoothly, and then ω is a contact form on M supported by (F, φ).
- By [2], there is the homeomorphism $b : D \to D$ such that
$$b \circ p = p \circ h.$$
In a similar way, we can construct a contact form ω_0 on S^3 suppored by the (trivial) open-book decomposition of S^3 with fibers $D \times t$ and the monodromy map b.

Proposition 2.2. Let (M, ξ) be a contact structure on M supported by (F, φ). Then there is a closed braid $\widehat{b} \subset S^3$ with axis L, and a simple cover $p : M \to S^3$

branched over $\widehat{b}$ such that $F \times t = p^{-1}(D \times t)$ for each $t \in [0,1]$. A contact form $\omega = p^*\omega_0$ defines a contact structure ξ' on M which is isotopic to ξ.

3. Results

Theorem 3.1. Let (M, ξ) be a contact structure supported by an open-book decomposition (F, φ) of M, and c a simple closed curve on F. There is a positive contact structure ξ' on M isotopic to ξ such that c is a Legendrian curve in (M, ξ') if and only if c is not null-homologous.

Corollary 3.2. Let K be a fibered knot in S^3 with a fiber surface F and a fibration φ. If there is a non-seperating loop c on F such that a tublar neighbourhood of c in F is an unknoted, untwisted annulus in S^3, then $\xi_{(F,\varphi)}$ is overtwisted.

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On the configuration space of points and the Casson invariant
(点の配置空間とキャッソン不変量について)

森山 哲裕 (Tetsuhiro Moriyama) [1]

2004/1/31, 13:30–18:00

Abstract. 三次元ホモロジー球面の点の配置空間と接束の自明化を用いてある４次元多様体を構成します．主定理は，その符号数がキャッソン不変量に比例する，というものです．前半はおおまかな構成を述べ，後半は結果の証明の解説をします．
In this talk, we will construct certain 4-manifold X by using the configuration space of points of an oriented closed homolology 3-sphere M and a trivialization of TM. The main theorem is that the signature of X is equal to the Casson invariant of M (up to multiplication by a constant).

Organization

This note was made form the resume written by the speaker together with certain additions and modifications based on the note taken by K. Ichihara. Added or largely modified parts are Section 2, 3, 6, and Subsection 4.1.

1. Introduction

Let M be an oriented closed homology 3-sphere, and $\lambda(M)$ Casson invariant of M. In this report, we construct some topological invariant $I(M)$ such that

- $I(M) = \lambda(M)$ (Theorem 1).
- $I(M) = -\frac{\operatorname{Sign} X_{f_M}}{8}$, where X_{f_M} is a certain 4-dimensional submanifold embedded in the two point configuration space of $M \setminus \{p\}$ (Theorem 2)

And we will also see outline of the proofs of these theorems (§ 8).

Roughly speaking, $\lambda(M)$ is defined by

$$\lambda(M) = \frac{1}{2} \# \frac{\operatorname{Hom}(\pi_1(M), SU(2))^{\mathrm{irr}}}{\text{conjugacy}},$$

and it is known that $\lambda(M)$ is determined by the Dehn surgery formula (c.f. [1]). On the other hand, $\lambda(M)$ is the only one non-trivial invariant which is finite type of degree 1 for both the algebraically split link surgery and Torelli surgery.

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Keywords and Phrases: Casson invariant, signature, configuration space, framing, finite type invariant.

Let X be an oriented compact smooth spin 4-manifold with boundary $\partial X = M$, such X always exists. Rohlin invariant $\mu(M)$ of M is defined by

$$\mu(M) \equiv \frac{\operatorname{Sign} X}{8} \pmod 2,$$

and there is a formula

$$\lambda(M) \equiv \mu(M) \pmod 2.$$

Namely, $\lambda(M)$ is an integral lift of Rohlin invariant $\mu(M)$.

On the face of things, the definitions of Casson invariant and Rohlin invariant looks very different: one is come from the flat $SU(2)$ connections, and the another one is from 4-dimensional. But some relations between Casson invariant and the signature of 4-manifolds are known. Here are such two examples as follows. Let $V(a_1, \ldots, a_n)$ be the Milnor fiber of the Seifert homology 3-sphere $\Sigma(a_1, \ldots, a_n)$, then

$$-\frac{\operatorname{Sign} V(a_1, \ldots, a_n)}{8} = \lambda(\Sigma(a_1, \ldots, a_n))$$

([7],[8]). The another example is about Ohta's invariant $\tau(M)$ (preprint). He constructed some oriented compact smooth 4-manifold N with boundary $\partial N \cong M$ in the moduli space of anti-self dual connections on some principal $SU(2)$ bundle $P \to M \times S^1$, and proved that N is spin and

$$\tau(M) = -\frac{\operatorname{Sign} N}{8}$$

is an topological invariant of M. Hence, one can see that

$$\tau(M) \equiv \mu(M) \pmod 2.$$

It is unknown if $\tau(M) = \lambda(M)$.

As mentioned above, this report gives some topological construction of Casson invariant by using the configuration spaces of 3-manifolds. This results depends on the work by Kuperberg-Thurston [11] that relate our invariant to Casson invariant. Our construction corresponds to the first non-trivial term of their invariant. In [11], they gave a purely topological definition of the perturbative quantum invariants of links and 3-manifolds. Ordinarily, this kind of work for the definition of the perturbative quantum invariants of 3-manifolds and links is by Kontsevich[10]. The related works, which uses the configuration spaces, was given by Axelrod-Singer [2, 3], Bott-Taubes [6], later by Bott-Cattaneo [4, 5], and Kuperberg-Thurston.

2. Basic definitions

In this section we recall basic definitions and notations. Please refer ... for these standard matters.

2.1. **Spin structure.** We start with the definition of the signature of 4-manifolds.

Let X^4 be a closed orientable 4-manifold and $Q : H^2(X,\mathbb{Z})/_{\mathrm{Tor}} \times H^2(X,\mathbb{Z})/_{\mathrm{Tor}} \to \mathbb{Z}$ the intersection form. Then Q is a symmetric, nondegenerate unimodular bilinear form. Set b_+ (resp. b_-) be the number of positive (resp. negative) eugenvalues of Q.

Definition 2.1 (the signature of X)**.** $\operatorname{Sign} X := b_+ - b_-$

Example 2.2.

- $X = \mathbb{C}P^2$. Then $Q = 1$ (i.e. $x \cdot x = 1$ for $\forall x$), and so, $\operatorname{Sign}(Q) = \operatorname{Sign}(1) = 1$.
- $X = S_1^2 \times S_2^2$. Then $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and so, $\operatorname{Sign}(S_1^2 \times S_2^2) = 1$.
- $X = K^3$ surface (or Kummer surface). Let

$$E_8 = \begin{pmatrix} -2 & & & 1 & & & & \\ & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & & \\ 1 & & 1 & -2 & 1 & & & \\ & & & 1 & -2 & 1 & & \\ & & & & 1 & -2 & 1 & \\ & & & & & 1 & -2 & \\ & & & & & & 1 & -2 \end{pmatrix}$$

and then, $Q = E_8 \oplus E_8 \oplus 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Thus $\operatorname{Sign}(X) = b_+ - b_- = -6$.

Concerning the signature of 4-manifolds, one of the most famous theorem is:

Theorem (Rohlin)**.** *If X is smooth and spin, then* $\operatorname{Sign}(X) \equiv 0 \mod 16$.

A spin 4-manifold is defined as follows.

Let $\pi : E \to X$ be a vector bundle over a smooth manifold X and $\{U_\lambda\}$ an open covering of X; $X = \bigcup_\lambda U_\lambda$. By definition of a vector bundle, $E \supset \pi^{-1}(U_\lambda) \cong U_\lambda \times \mathbb{R}^n$ (n denotes the dimension of the fiber) and there exists $g_{_{UV}} : U \cap V \to GL(n,\mathbb{R})$ satisfying $g_{_{UV}} g_{_{VW}} g_{_{WV}} = 1$ for $U \cap V \cap W \neq \emptyset$ (called the *cocycle condition*). Up to homotopy we may assume that each $g_{_{UV}} \in SO(n) \subset GL(n,\mathbb{R})$.

Recall that $\pi_1(SO(n)) \cong \mathbb{Z}_2$ for $n \geq 3$. We define $Spin(n)$ by the double covering $\pi : Spin(n) \overset{\mathbb{Z}_2}{\to} SO(n)$.

Definition 2.3 (spin structure on a vector bundle E, (I))**.** A *spin structure* σ on E is defined as $\sigma = \{\widetilde{g_{UV}}\}$ such that $\widetilde{g_{UV}} : U \cap V \to Spin(n)$ with $\pi(\widetilde{g_{UV}}) = g_{UV}$ and $\widetilde{g_{UV}}\widetilde{g_{VW}}\widetilde{g_{WV}} = 1 \in Spin(n)$.

Definition 2.4 (spin structure on a vector bundle E, (II))**.** A *spin structure* σ on E is defined as a spin structure σ on $E \oplus \mathbb{R}^N$ for some $N \geq 2$.

It is known that these two definitions make no contradiction.

Definition 2.5. A smooth manifold X is *spin* if its tangent bundle admits a spin structure.

Here we give some remarks. Given $g_{UV} : U \cap V \to GL(n, \mathbb{R})$ and $\pi_1(SO(n)) \cong \mathbb{Z}_2$, a local lift $\widetilde{g_{UV}} : U \cap V \to Spin(n)$ always exists. For such local lifts, set $h_{UVW} := \widetilde{g_{UV}}\widetilde{g_{VW}}\widetilde{g_{WV}}$. Then $\{h_{UVW}\}$ gives a cocycle in $C^2(X, \mathcal{U}, \mathbb{Z}_2)$ with some open covering $\mathcal{U}$ of X. This cocycle represents the cohomology class in $H^2(X, \mathcal{U}, \mathbb{Z}_2)$, which is equal to the Stiefel-Whitney class w_2. Thus we have the following:

Fact. *On a vector bundle $\pi : E \to X$ over a smooth manifold X, there exists a spin structure if and only if the Stiefel-Whitney class $w_2 \in H^2(X, \mathbb{Z}_2)$ is zero.*

From this fact, we can observe that a spin structure σ on a vector bundle E over X gives a 'trivialization' next to the 'orientation'. That is; by using σ, one can have $E|_{X^{(2)}} \cong X^{(2)} \times \mathbb{R}^n$, where $X^{(2)}$ denotes the 2-skeleton of X endowed with a CW-complex structure.

2.2. **Rohlin invariant.** In this subsection we introduce Rohlin invariant of 3-manifolds.

Let M^3 be an oriented closed homology 3-sphere (i.e., $H_*(M, \mathbb{Z}) = H_*(S^3, \mathbb{Z})$). Note that

- M is spin, for $w_2 \in H^2(TM, \mathbb{Z}_2) = 0$, and
- the spin structure on TM is unique up to homotopy, for the difference $diff(\sigma_1, \sigma_2)$ of the spin structures σ_1, σ_2 on TM lies in $H^1(X, \mathbb{Z}_2)$, which actually vanishes for a homology 3-sphere M.

Moreover we have the following facts.

Fact.

(1) There exists an oriented compact smooth spin simply-connected 4*-manifold X such that $\partial X = M$ (originally due to Thom).*

(2) *For such X, the intersection form $Q : H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to \mathbb{Z}$ is non-degenerate, and* $\operatorname{Sign}(Q) \equiv 0 \mod 8$*, i.e.,* $\frac{\operatorname{Sign}(Q)}{8} \in \mathbb{Z}$*. This follows from the facts that M is spin and an algebraic property of Q: Q is an even form.*

Now we define:

Definition 2.6 (Rohlin invariant $\mu(M)$ of M)**.** $\mu(M) := \frac{\operatorname{Sign}(X)}{8} \in \mathbb{Z}_2$

This $\mu(M)$ is well-defined: It suffice to show that it is independent from the choice of the 4-manifold with boundary M. Let X' be another 4-manifold with $\partial X' = M$. By the Novikov additivity, $\operatorname{Sign}((-X') \cup X) = -\operatorname{Sign}(X') + \operatorname{Sign}(X)$ holds, where $-X'$ denotes a copy of X' with opposite orientation. This $\operatorname{Sign}((-X') \cup X)$ have to be zero modulo 16 by the Rohlin's theorem, since $(-X') \cup X$ is a closed smooth spin 4-manifold. It concludes that $\operatorname{Sign}(X') \equiv \operatorname{Sign}(X) \mod 16$, and so, $\frac{\operatorname{Sign}(X)}{8} = \frac{\operatorname{Sign}(X')}{8} \in \mathbb{Z}_2$.

2.3. **Casson invariant.** In this subsection we give some facts and a conjecture about Casson invariant, which is a motivation of my work.

As in Section 1, the original definition (due to Casson) of Casson invariant $\lambda(M)$ is given, very roughly, by

$$\lambda(M) = \frac{1}{2} \# \frac{\operatorname{Hom}(\pi_1(M), SU(2))^{\text{irr}}}{\text{conjugacy}}.$$

This is shown to be an integer-valued topological invariant for oriented closed homology 3-spheres.

In present, another definition is also known. This is axiomatical; $\lambda(M)$ is determined by the following inductively.

- $\lambda(S^3) = 0$.
- $\lambda(-M) = \lambda(M)$.
- $\lambda(M_1 \# M_2) = \lambda(M_1) + \lambda(M_2)$ for the connected sum $M_1 \# M_2$.
- $\lambda(M_K) - \lambda(M) = \frac{1}{2}\Delta''_K(1)$ (Dehn surgery formula).

See [1] for example.

As we stated in Section 1, Casson invariant $\lambda(M)$ can be regarded as an integral lift of Rohlin invariant $\mu(M)$: that is, a formula

$$\lambda(M) \equiv \mu(M) \pmod 2$$

is known.

Since Rohlin invariant is defined as $\mu(M) \equiv \frac{1}{8}\operatorname{Sign}(X) \mod 2$ for some 4-manifold X with boundary M, it might be possible that one can find a 4-manifold

X with boundary M such that $\lambda(M) = \frac{1}{8}\mu(M)$. Concerning this observation, the following conjecture is known.

Conjecture (Casson Invariant Conjecture (Neumann))**.** *Signature of (some special) Milnor fiber of M is equal to $8\lambda(M)$.*

See *** for detail about this conjecture. Some partial positive answers to the conjecture have been obtained.

Theorem (Fintushel and Stern [7])**.** *Consider $f(X,Y,Z) = X^p + Y^q + Z^r$ with p,q,r coprime integers. Let $V(f) := \{(X,Y,Z) \in \mathbb{C}^3 \mid f(X,Y,Z) = 0\} \subset \mathbb{C}^3$ and $\Sigma(p,q,r) := V(f) \cap S^5$. Here we regard S^5 as the unit sphere in $\mathbb{R}^6 = \mathbb{C}^3$. It is shown that the map*

$$\frac{f}{|f|} : S^5 \setminus \Sigma(p,q,r) \to S^1$$

gives a fiber bundle structure. After suitable compactification, the fiber (called Milnor fiber) is regarded as a 4-manifold X with $\partial X = \Sigma(p,q,r)$. Then

$$\lambda(\Sigma(p,q,r)) = \frac{\operatorname{Sign} X}{8}$$

holds.

In other words, for $\Sigma(p,q,r)$, the signature of a Milnor fiber X is a topological invariant which is an integral lift of Rohlin invariant.

3. Idea

What we see in this section is the background idea to get an invariant which is an integral lift of Rohlin invariant. To do this, we use the *two point configuration space* of a 3-manifold. See the next section for precise definitions of the terms in the following.

3.1. **Finding X^4_f.** Let (M,f) be a 3-manifold M and a framing f of the 'punctured' M. Consider the two point configuration space

$$C_2(\hat{M}) := M \times M \setminus (* \times M \cup M \times * \cup \{(x,x)\})$$

of 'punctured' M, which is assumed to be compactified 'suitably'. Thus this is a compact 6-manifold with non-empty boundary.

We will construct a 'partial Gauss map' $\varphi_f : U \to S^2$, where U is a complement of a certain compact subset of $C_2(\hat{M})$ defined by using f. Intuitively this map means

$$M \times M \ni (x,y) \mapsto \frac{y-x}{||y-x||} \in S^2.$$

By the way of compactification, this map naturally extends to the map of $\partial C_2(\hat{M})$.

Now we suppose the existence of a map

$$\tilde{\varphi}_f \colon C_2(\hat{M}) \to S^2$$

such that $\tilde{\varphi}_f|_{\partial C_2(\hat{M})} = \varphi_f$. Taking a regular value $v \in S^2$ of $\tilde{\varphi}_f$, set

$$X_f = \tilde{\varphi}_f^{-1}(v).$$

Then X_f is an oriented compact smooth 4-manifold satisfying

(1) $\partial X_f \cong M \# M \# (-M)$,

(2) X_f is spin,

(3) the signature σ_f of X_f depends only on (M, f).

The property (1) is from the definition of φ_f. Since the normal bundle of $X_f \subset C_2(\hat{M})$ and the tangent bundle $TC_2(\hat{M})$ are spin, X_f is also spin, so we have (2). Another manifold $X_f{}'$ come from another map $\tilde{\varphi}'_f \colon C_2(\hat{M}) \to S^2$ is cobordant to X_f relative to the boundary, hence. Consequently, σ_f is an invariant of a pair (M, f), and we obtain (3). By the definition of Rohlin invariant, one can see

$$\frac{\sigma_f}{8} \equiv \mu(M) \pmod 2.$$

Therefore, we get an integral lift $\sigma_f/8$ of Rohlin invariant.

3.2. **Finding $\tilde{\varphi}_f$.** In this subsection we explain an idea to find $\tilde{\varphi}_f \colon C_2(\hat{M}) \to S^2$ such that $\tilde{\varphi}_f|_{\partial C_2(\hat{M})} = \varphi_f$.

Suppose that φ_f as before is already given. We want to apply 'obstruction theory'.

Lemma. *we have*

$$H^i(C_2(\hat{M}), \partial C_2(\hat{M})) \cong H^i(M \times M, A) = \begin{cases} \mathbb{Z} & i = 4, 6 \\ 0 & \textit{otherwise} \end{cases}$$

where $A = * \times M \cup M \times * \cup \{(x, x)\}$.

This follows from

$$H^i(M \times M) = \begin{cases} \mathbb{Z} & i = 1 \\ \mathbb{Z}^2 & i = 4, 6 \\ 0 & \text{otherwise} \end{cases}$$

and

$$H^i(A) = \begin{cases} \mathbb{Z} & i = 1 \\ \mathbb{Z}^3 & i = 4 \\ 0 & \text{otherwise} \end{cases}.$$

By the lemma above the primary obstruction class $o^4(\varphi_f)$ lies in

$$H^4(C_2(\hat{M}), \partial C_2(\hat{M}); \pi_3 S^2) \cong H^4(C_2(\hat{M}), \partial C_2(\hat{M}); \mathbb{Z}) \cong \mathbb{Z}.$$

If we can establish $o^4(\varphi_f) = 0$, i.e., we obtain $\varphi_f^{(4)} : X^{(4)} \to S^2$ such that $\varphi_f^{(4)}|_{\partial C_2(\hat{M})} = \varphi_f$, where $X^{(4)}$ denotes the 4-skeleton of $(C_2(\hat{M}), \partial C_2(\hat{M}))$, the secondary obstruction class $o^6(\varphi_f^{(4)})$ lies in

$$H^4(C_2(\hat{M}), \partial C_2(\hat{M}); \pi_5 S^2) \cong H^4(C_2(\hat{M}), \partial C_2(\hat{M}); \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

We will in fact achieve $o^4(\varphi_f) = 0$. However, unfortunately, we cannot get rid of $o^6(\varphi_f^{(4)})$, and so, we will take another way to construct our invariant.

4. Configuration space of points and Gauss map

In this section we first introduce the two point configuration space of a 'punctured' 3-manifold and compactify it. Then we define a 'partial Gauss map' φ_f as explained in the previous section.

4.1. $C_2(\hat{M})$ and $\partial C_2(\hat{M})$.

The *two point configuration space* $C_2(X)$ of a space X is defined by

$$C_2(X) = \{(x, y) \mid x, y \in X,\ x \neq y\}.$$

Let M be an oriented closed homology 3-sphere and set $\hat{M} = M \# \mathbb{R}^3$. We compactify $C_2(\hat{M})$ by the method of Bott-Taubes's paper.

Let $\Delta^3 := \{(x, x, x)\} \subset M \times M \times M$. This can be regarded as an image of a smooth embedding of M into $M \times M \times M$. Also let $\Delta^2 := \{(x, x, y), (x, y, x), (y, x, x)\} \subset M \times M \times M$. Obviously Δ^3 is a subset of Δ^2. Thus $\Delta^2 \cap (M \times M \times M \setminus N_3)$ is smoothly embedded into $M \times M \times M \setminus N_3$, where N_3 denotes an open neighborhood of Δ^3. Let N_2 denote an open neighborhood of $\Delta^2 \cap (M \times M \times M \setminus N_3)$. Set $C_3^c(M) := \times M \times M \setminus N_2 \setminus N_3$, which in fact gives a compactification of $C_3(M)$.

Now we define a compactification $C_2^c(\hat{M})$ of $C_2(\hat{M})$ as the following diagram commutes.

$$\begin{array}{ccc} C_2^c(\hat{M}) & \longrightarrow & C_3^c(M) \ni (x, y, z) \\ \downarrow & & \downarrow \\ \{p\} & \longrightarrow & M \ni z \end{array}$$

This compactification gives a homotopy equivalence between $C_2^c(\hat{M})$ and $C_2(\hat{M})$. In the following we abuse $C_2(\hat{M})$ to also denote $C_2^c(\hat{M})$.

4.2. **The map** $\varphi_f \colon \partial C_2(\hat{M}) \to S^2$**.** In this subsection, we construct a "partial Gauss map" $\varphi_f \colon U \to S^2$, where U is a complement of a certain compact subset of $C_2(\hat{M})$ by using some additional data f. By the definition of the compactification of $C_2(\hat{M})$, φ_f defined on U naturally extends to $\partial C_2(\hat{M})$, which gives a fiber bundle structure of $\partial C_2(\hat{M})$ over S^2.

At first we give φ_f for the Euclidean space $\mathbb{R}^3$ as an easiest, but instructive example. Let

$$\varphi_{\mathbb{R}^3} \colon C_2(\mathbb{R}^3) \to S^2$$

be the map defined by

$$\varphi_{\mathbb{R}^3}(x, y) = \frac{y - x}{\|y - x\|}$$

for $(x, y) \in C_2(\mathbb{R}^3)$ which is called Gauss map. There exists a homeomorphism

$$C_2(\mathbb{R}^3) \cong \mathbb{R}^3 \times (0, \infty) \times S^2, \quad (x, y) \mapsto (z, r, v)$$

by the corresponding

$$z = \frac{x + y}{2}, \quad r = \frac{\|y - x\|}{2}, \quad v = \varphi_{\mathbb{R}^3}(x, y).$$

Immediately, the pre-image ${\varphi_{\mathbb{R}^3}}^{-1}(v)$ of a point $v \in S^2$ is contractible. In particular, its signature is zero. As we will see later (Theorem 2), this vanishing corresponds to the fact $\lambda(S^3) = 0$.

Let M be an oriented closed homology 3-sphere and set $\hat{M} = M \# \mathbb{R}^3$. Let $f \colon T\hat{M} \to \hat{M} \times \mathbb{R}^3$ be a framing of $T\hat{M}$. In this report, we always assume that any framing of $T\hat{M}$ is compatible with Euclidean framing $f_{\mathbb{R}^3}$ over the complement of some compact subset of $\hat{M}$. Now, we define a map $\varphi_f \colon U \to S^2$ as follows, where $U = U_1 \cup U_2 \cup U_3$. Each U_i is defined in later three cases.

4.2.1. *Two points are very close.* First, let

$$U_1 = \{(x, y) \in C_2(\hat{M}) \mid d(x, y) < \varepsilon\},$$

where d is a metric of $\hat{M}$ and $\varepsilon > 0$ is small enough. Suppose $(x, y) \in U_1$. Then we can define the "direction" $\varphi_f(x, y) \in S^2$ from x to y by using f.

4.2.2. *one point is in the end of* $\hat{M}$. Suppose that $\hat{M}$ is obtained by connecting a 3-disk $D_p \subset M$ around $p \in M$ and $D^3 \subset \mathbb{R}^3$ around 0. Let $V \subset \hat{M}$ be the open subset corresponding to $M \setminus D_p$, and $W \subset \hat{M}$ corresponding $\mathbb{R}^3 \setminus D^3$. Let

$$U_2 = (V \times W) \cup (W \cup V).$$

We assume that f is coincides with $f_{\mathbb{R}^3}$ on some open set including the closure of W.

If $x \in V$ and $y \in W$, then define $\varphi_f(x,y) = y$. Note that this definition makes sense, because we can think y as a point in $\mathbb{R}^3$. Also define $\varphi_f(y,x) = -y$.

4.2.3. *Both two points in the end.* Let

$$U_3 = C_2(W),$$

and suppose $(x,y) \in U_3$. In this case, we define

$$\varphi_f(x,y) = \frac{y-x}{\|y-x\|}.$$

From the way of compactification, we have the following.

Lemma. *The map $\varphi_f \colon U \to S^2$ naturally extends to $\varphi_f \colon \partial C_2(\hat{M}) \to S^2$* (we abuse the notation φ_f). *The fiber $\varphi_f^{-1}(p)$ for every regular value $p \in S^2$ of $\varphi_f \colon \partial C_2(\hat{M}) \to S^2$ is diffeomorphic to $M \# M \# (-M)$.*

5. Two numbers d_f and σ_f

In this section, we define two integers d_f and σ_f used to define the invariant $I(M)$.

Let

$$\bar{C}_2(\hat{M}, f) = S^2 \cup_{\varphi_f} C_2(\hat{M})$$

be the attaching space by the map $\varphi_f \colon \partial C_2(\hat{M}) \to S^2$. Using the long exact sequence of $(\bar{C}_2(\hat{M}, f), S^2)$, we have an isomorphism

$$H^k(\bar{C}_2(\hat{M}, f); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, 6 \\ 0, & \text{otherwise.} \end{cases}$$

The following lemma define the integer d_f, this definition is similar to the cohomological definition of the Hopf invariant(c.f. [9]).

Lemma 5.1. *There exists a graded ring isomorphism*

$$H^*(\bar{C}_2(\hat{M}, f); \mathbb{Z}) \cong \mathbb{Z}[a, b]/(a^2 - d_f b, b^2)$$

for some integer d_f, where $\deg a = 2$, $\deg b = 4$.

This number d_f is nothing but the Casson invariant for the framed 3-manifold ([11]).

Let $L_{S^2} \to S^2$ be a complex line bundle with Euler number 1, and s_{S^2} a generic section. We can assume that ${s_{S^2}}^{-1}(0)$ consists just one point. Set $L_f = {\varphi_f}^* L_{S^2}$, $s_f = {\varphi_f}^* s_{S^2}$. Then $s_f : \partial C_2(\hat{M}) \to L_f$ is a generic section, and ${s_f}^{-1}(0) \cong M \# M \# (-M)$. Since the inclusion $\partial C_2(\hat{M}) \hookrightarrow C_2(\hat{M})$ induces an isomorphism on

H^2, there exists only one isomorphism class of a complex line bundle $\tilde{L}_f \to C_2(\hat{M})$ such that $\tilde{L}_f|_{\partial C_2(\hat{M})} = L_f$.

$$\begin{array}{ccc} L_f & \longrightarrow & \tilde{L}_f \\ \big\downarrow & & \big\downarrow \\ \partial C_2(\hat{M}) & \longrightarrow & C_2(\hat{M}) \end{array}$$

Let $\tilde{s}_f : C_2(\hat{M}) \to \tilde{L}_f$ be a generic section such that $\tilde{s}_f|_{\partial C_2(\hat{M})} = s_f$. Let

$$X_f = \tilde{s}_f^{-1}(0),$$

then X_f is an oriented compact smooth 4-manifold with boundary $\partial X_f \cong M\#M\#(-M)$. Define

$$\sigma_f = \operatorname{Sign} X_f.$$

Another choices $\tilde{s}'_f$ give the same value of σ_f, because a generic homotopy between $\tilde{s}_f$ and $\tilde{s}'_f$ gives a cobordism between X_f and X'_f.

Definition 5.2.

$$I(M) = \frac{d_f - \sigma_f}{8}$$
$$I(M, f) = (I(M), d_f)$$

6. Legitimacy of invariants

We have just defined our 'invariant' $I(M)$, which will be shown to be an integer-valued, topological invariant of M (Section 8) and to be equal to Casson invariant of M (Section 9).

In this section, we try to give another 'definition' of $I(M, f)$, which could explain the legitimacy of $I(M, f)$. The idea behind the explanation was given in Subsection 3.2. Please remark that this section contains some unreliable arguments.

Now we have the map $\varphi_f : \partial C_2(\hat{M}) \to S^2$ and the attaching space $\bar{C}_2(\hat{M}, f) = S^2 \cup_{\varphi_f} C_2(\hat{M})$ by φ_f. This $\bar{C}_2(\hat{M}, f)$ is smooth except on S^2. In fact, the 'normal disk' at $p \in \bar{C}_2(\hat{M}, f) \setminus S^2$ is regarded as a cone over a generic fiber $\varphi_f^{-1}(p) \cong M\#M\#(-M)$. Here we assume that:

Assumption. *The attaching space $\bar{C}_2(\hat{M}, f)$ is smooth everywhere.*

Let $\bar{L}_f \to \bar{C}_2(\hat{M}, f)$ be the complex line bundle naturally obtained from $\tilde{L}_f \to C_2(\hat{M})$ and $L_{S^2} \to S^2$, and then, consider the spin bordism class of $\bar{L}_f \to \bar{C}_2(\hat{M}, f)$.

The *n-dimensional spin bordism group* is defined by

$$\Omega_n^{spin}(X) := \left\{ (Z,h) \;\middle|\; \begin{array}{l} Z: \text{ spin closed smooth } n\text{-manifold} \\ h: Z \to X, \text{ continuous map} \end{array} \right\} \Big/ \text{cobordant}$$

Here (Z_1, h_1) is said to be *cobordant* to (Z_2, h_2) if there exists a spin $(n+1)$-manifold W and a continuous map $\tilde{h}\colon W \to X$ such that $\partial W = Z_1 \amalg (-Z_2)$ and $\tilde{h}|_{\partial W} = h_1 \amalg h_2$.

Example 6.1. Define $\Omega_n^{spin} := \Omega_n^{spin}(*)$. Then the following table is known.

n	0	1	2	3	4	5	6	7
Ω_n^{spin}	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0

In detail:

- The generator of $\Omega_1^{spin} \cong \mathbb{Z}_2$ given by Lie framing, i.e., the framing obtained from a framing at one point by distributing by the elements of the Lie group S^1.
- $\Omega_3^{spin} = 0$ indicates that there always exists an oriented compact spin 4-manifold bounded by given closed 3-manifold.
- The isomorphism $\Omega_4^{spin} \cong \mathbb{Z}$ is given by the correspondence $X^4 \mapsto \frac{\operatorname{Sign} X}{16}$.

In the case of $n = 6$ and $X = \mathbb{C}P^\infty$ we have the following.

Proposition. $\Omega_6^{spin}(\mathbb{C}P^\infty) \cong \mathbb{Z} \oplus \mathbb{Z}$

The group $\Omega_n^{spin}(\mathbb{C}P^\infty)$ is called the *n-dimensional spin bordism group decorated with a complex line bundle*, for there exists a correspondence between the complex line bundles over Z and homotopy classes of continuous maps of Z to $\mathbb{C}P^\infty$. Using this correspondence, the isomorphism in the proposition above is given by

$$[L \to Z] \mapsto \left(\frac{d - \sigma}{8}, d \right)$$

where Z denotes a 6-manifold, L a complex line bundle over Z, $d := \int_Z c_1(L)^3$ with $c_1(L)$ the first Chern form, and $\sigma := \operatorname{Sign} s^{-1}(0)$ with $s\colon Z \to L$ a generic section.

In our setting $Z = \bar{C}_2(\hat{M}, f)$ and $L = \bar{L}_f$, this value $\left(\frac{d-\sigma}{8}, d\right)$ is actually coincident with $I(M, f) = (\frac{d_f - \sigma_f}{8}, d_f)$.

7. Statements and Examples

Let (M, f) be an oriented closed homology 3-sphere with a framing of $T\hat{M}$. Let $I(M)$ be the number defined in Definition 5.2.

Theorem 1 ([14])**.** *$I(M)$ is a $\mathbb{Z}$-valued topological invariant of M, and it equals to Casson invariant of M.*

Now, we see some examples of the calculations of $I(M)$ and $I(M,f)$.

7.1. **Case of $M = S^3$ with Euclidean framing.** Let $M = S^3$, then $\hat{M} = \mathbb{R}^3$. Let $f_{\mathbb{R}^3}$ be the Euclidean framing on $T\mathbb{R}^3$, and we have $\varphi_{f_{\mathbb{R}^3}} = \varphi_{\mathbb{R}^3}$.

One can take the line bundle $\tilde{L}_{f_{\mathbb{R}^3}} \to C_2(\hat{M})$ as the pull-back bundle $\varphi_{f_{\mathbb{R}^3}}{}^* L_{S^2}$, and the pull-back section

$$\tilde{s}_{f_{\mathbb{R}^3}} = \varphi_{\mathbb{R}^3}^* s_{S^2} \colon C_2(\hat{M}) \to \tilde{L}_{f_{\mathbb{R}^3}}$$

is generic. If $v = s_{S^2}^{-1}(0)$, then we have

$$X_{f_{\mathbb{R}^3}} = \tilde{s}_{f_{\mathbb{R}^3}}^{-1}(0) = \varphi_{\mathbb{R}^3}^{-1}(v) \cong \mathbb{R}^4,$$

and so, we obtain $\sigma_{f_{\mathbb{R}^3}} = 0$.

Since $\bar{C}_2(\mathbb{R}^3) \cong S^4 \times S^2$, we have the ring isomorphism

$$H^*(\bar{C}_2(\hat{M}, f); \mathbb{Z}) \cong R_0,$$

this implies $d_{f_{\mathbb{R}^3}} = 0$. Therefore, we have

$$I(S^3) = \frac{d_{f_{\mathbb{R}^3}} - \sigma_{f_{\mathbb{R}^3}}}{8} = \frac{0-0}{8} = 0$$

and

$$I(S^3, f_{\mathbb{R}^3}) = (I(M), d_{f_{\mathbb{R}^3}}) = (0,0).$$

7.2. **Case of $M = S^3$ with any framings.** Let f be an any framing of $\mathbb{R}^3$ which is always obtained by $f = g f_{\mathbb{R}^3}$ for some compact supported map $g \colon \mathbb{R}^3 \to SO(3)$. Any framings on $\mathbb{R}^3$ are classified by the degree $\deg g \in \mathbb{Z}$ of the induced homomorphism $g^* \colon H_c^3(SO(3); \mathbb{Z}) \to H_c^3(\mathbb{R}^3; \mathbb{Z})$.

Let $n = \deg g$. The map $\varphi_f \colon \partial C_2(\hat{M}) \cong S^3 \times S^2 \to S^2$ essentially equals to the evaluation map ev_g as follows:

$$ev_g \colon S^3 \times S^2 \to S^2, \quad (x,v) \mapsto g(x)v$$

Therefore, we have $\bar{C}_2(\hat{M}, f) \cong S(E_n)$, where $\pi \colon E_n \to S^4$ is a real vector bundle with $\langle [S^4], p_1(E_n) \rangle = 4n$ and $S(E_n)$ the associated sphere bundle. Calculating the characteristic classes of $TS(E_n)$ and $\pi^* E_n$([13]), we obtain that $d_f = n$ and $\sigma_f = n$. This implies that

$$I(S^3, f) = (\frac{d_f - \sigma_f}{8}, d_f) = (0, n),$$

and of course, we obtain $I(S^3) = 0$ again.

7.3. **Connected sum.** Let (M_1, f_1), (M_2, f_2) be framed manifolds. Set $M = M_1 \# M_2$, $f = f_1 \# f_2$. Let us think $\hat{M}_1 = M_1 \# \mathbb{R}_1$, $\hat{M}_2 = M_2 \# \mathbb{R}_2$, where

$$\mathbb{R}_1 = \{(x_1, x_2, x_3) \mid x_1 < 0\}, \quad \mathbb{R}_2 = \{(x_1, x_2, x_3) \mid x_1 > 0\},$$

and each f_i is a faming over $\hat{M}_1$ compatible with the Euclidean framing on the end. Moreover, we suppose that $\hat{M}_i$ is the connected sum at around a point in M_i and around $(\pm R, 0, 0) \in \mathbb{R}_i$ for some large number $R \gg 1$. Then we can take $\hat{M}$ as

$$\hat{M}_1 \cup V \cup \hat{M}_2$$

such that the M_1-part and M_2-part in $\hat{M}$ are very far each other, where $V = (-1, 1) \times \mathbb{R}^2$.

Next, we will construct $\varphi_f \colon \partial C_2(\hat{M}) \to S^2$. Define a map

$$\varphi_{ij} \colon (V \cup \hat{M}_i) \times (V \cup \hat{M}_j) \setminus \Delta \to S^2$$

as follows ($i, j = 1, 2$, $i \neq j$). Let $h \colon \hat{M} \to \mathbb{R}^3$ be the map obtained by collapsing each M_i-part to $(\pm R, 0, 0)$. Then $\varphi_{ij}(x, y)$ is defined by

$$\varphi_{ij}(x, y) = \frac{h(y) - h(x)}{\|h(y) - h(x)\|}.$$

Let $\tilde{\varphi}_{f_i} \colon C_2(M_i) \to \mathbb{C}P^3$ be the classifying map of $\tilde{L}_{f_i}$. Let $\tilde{\varphi}_f \colon C_2(\hat{M}) \to \mathbb{C}P^3$ be an extension map of $\tilde{\varphi}_f$ obtained from $\tilde{\varphi}_{f_1}$, $\tilde{\varphi}_{f_2}$ and φ_{ij}. Note that any two such maps are coincide on these common domain. Let $\mathbb{C}P^{2'} \subset \mathbb{C}P^3$ be a submanifold homologous to $\mathbb{C}P^2$ that transversally intersect with $\mathbb{C}P^1 \cong S^2$ at one point $v = (1, 0, 0) \in S^2$. There exists a generic section of the complex line bundle $L_{\mathbb{C}P^3} \to \mathbb{C}P^3$ with $c_1(L_{\mathbb{C}P^3}) = 1$ such that the pre-image of zero is $\mathbb{C}P^{2'}$. Hence

$$X_f = \varphi_f(\mathbb{C}P^{2'}) \cong X_{f_1} \sharp X_{f_2},$$

this means $\sigma_f = \sigma_{f_1} + \sigma_{f_2}$. And also it is easy to see that $d_f = d_{f_1} + d_{f_2}$. Therefore, we obtain the following:

Proposition 7.1.

$$I(M, f) = I(M_1, f_1) + I(M_2, f_2)$$

$$I(M) = I(M_1) + I(M_2)$$

7.4. **Opposite orientation.** Let M' be M with the opposite orientation, and let $f' = (-f_1, f_2, f_3)$ where $f = (f_1, f_2, f_3)$. By the definition of d_f and σ_f, we have

$$d_{f'} = -d_f, \quad \sigma_{f'} = -\sigma_f.$$

This implies that

Proposition 7.2.

$$I(M', f') = -I(M, f)$$

$$I(M') = -I(M)$$

8. Outline of proof

8.1. **Integrality of $I(M)$.** Since $\Omega_5^{spin}(S^2) = 0$([15]), there exists an oriented compact smooth spin 6-manifold Z with a complex line bundle $L_Z \to Z$ such that

$$\partial Z = \partial C_2(\hat{M}), \quad L_Z|_{\partial C_2(\hat{M})} = L_f$$

and the image of the classifying map of L_Z is contained in S^2. Let

$$W = C_2(\hat{M}) \cup_{\partial C_2(\hat{M})} Z, \quad L_W = \tilde{L}_f \cup_{L_f} L_Z \to W.$$

Then, (W, L_W) is an oriented closed smooth spin 6-manifold with a complex line bundle. Applying the index theorem to (W, L_W), one can see that the integral

$$\int_W ch(L_W)\hat{\mathcal{A}}(TW)$$

is an integer(c.f. [12]). Here, ch is the Chern character and $\hat{\mathcal{A}}$ is the $\hat{\mathcal{A}}$-genus. This value equals to

$$I(M) - \frac{\operatorname{Sign} X_Z}{8},$$

where X_Z is the pre-image of 0 of a generic section, which is an extension of s_f of L_f, of L_Z. Since X_Z is spin and ∂X_Z is a homology 3-sphere, we have $\operatorname{Sign} X_Z \equiv 0 \pmod 8$. Therefore, we obtain the following proposition.

Proposition 8.1. *The number $I(M)$ is an integer.*

8.2. **Topological invariance of** $I(M)$**.** Let f, f' be framings of $T\hat{M}$. There exists a one-to-one correspondence between the set of homotopy classes of framings on $\hat{M}$ and $[\hat{M}, SO(3)]_c$, i.e., the set of maps with compact supports. Thus f' can be represented by $f' = gf$ for some $g\colon \hat{M} \to SO(3)$.

Then $(M, f') \cong (M\#S^3, f\#gf_{\mathbb{R}^3})$. According to Proposition 7.1,

$$I(M, f') = I(M, f) + I(S^3, gf_{\mathbb{R}^3}) = I(M, f) + (0, \deg g).$$

In other words,

$$d_{f'} = d_f + \deg g, \quad \sigma_{f'} = \sigma_f + \deg g.$$

In particular, $I(M)$ does not depend on f. Therefore

Proposition 8.2. *$I(M)$ is an topological invariant of M.*

8.3. **Casson invariant.** Kuperberg-Thurston showed that some value $\tilde{I}_1(M)$ is Casson invariant by using the theory of finite type invariants of homology 3-spheres in their paper [11]. The invariant $\tilde{I}(M)$ is constructed as follows. First, define

$$I_1(M) = \frac{1}{6}\langle \bar{C}_2(\hat{M}, f), c_1(L_f)^3\rangle,$$
$$\delta_1(M) = \frac{1}{24}\langle X_f, p_1(TC_2(\hat{M})|_{X_f})\rangle.$$

And then $I(M)$ is defined by

$$\tilde{I}_1(M) = I_1(M) - \delta_1(M).$$

Calculating characteristic classes, we have the following proposition.

Proposition 8.3. $\tilde{I}_1(M) = I(M)$.

By Proposition 8.3, we obtain Theorem 1. To prove that $I(M)$ is a $\mathbb{Z}$-valued topological invariant, one only need Proposition 8.1,8.2.

9. Casson invariant as a signature

By § 8.2, we obtain the following proposition.

Proposition 9.1. *There exists only one framing f_M of $T\hat{M}$ such that $d_{f_M} = 0$.*

Therefore, we have

$$I(M) = \frac{d_{f_M} - \sigma_{f_M}}{8} = -\frac{\sigma_{f_M}}{8}.$$

Theorem 2.

$$\lambda(M) = -\frac{\sigma_{f_M}}{8}.$$

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A surgery description of homology solid tori and its applications to the Casson invariant

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Abstract. It is known that every homology solid torus can be obtained from the standard solid torus $D^2 \times S^1$ by surgery on a suitable boundary link. In particular, every knot K in S^3 has a surgery description $(k;(k_i, e_i)_{i=1,2,\ldots,m})$, where k is a trivial knot, (k_i, e_i) is an e_i-framed knot in $S^3 - k$ with $e_i = \pm 1$, which satisfies the following properties:

- $k_1, \ldots, k_m$ bound mutually disjoint genus one Seifert surfaces $S_1, \ldots, S_m$ in $S^3 - k$,
- there non-separating simple closed curves x_i, y_i on each S_i such that
  - x_i intersects y_i transversely in a single point,
  - x_i bounds a disk D_i with $D_i \cap S_i = \partial D_i = x_i$ which meets k in a single point,
  - the linking number $\mathrm{lk}(y_i, k) = 0$.
- $a_2(K) = -\sum_{i=1}^{m} e_i \mathrm{lk}(y_i, y_i^+)$.

Using such a surgery description for K, we study the relation between the Alexander invariant of K and the Casson-Walker-Lescop invariant of the cyclic covering spaces of S^3 branched along K.

1. Notation

M, H : an oriented 3-manifold, an oriented $\mathbb{Z}\mathrm{H}S^3$,

$K \subset H$: a knot in H,

$\mathcal{L}$ (resp. $\mathcal{K} = (K, \gamma)$): a rational framed link (resp. knot) in H,

$X' = \chi(X; \mathcal{L})$: the object obtained from X by surgery along (K, γ), and we call $(X; \mathcal{L})$ a *surgery description for X'*,

lk : the linking number,

λ: the Casson invariant,

$\Delta_{K \subset H}(t)$: the symmetric Alexander polynomial of K in H,

$\varphi : M_K^r \to H$: the r-fold cyclic cover of H branched along a knot $K \subset H$.

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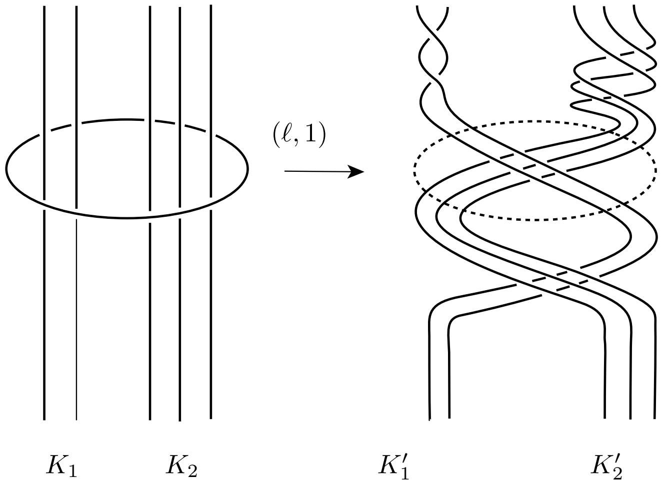
[2]See [20] for the details.

Lemma 1.1. Let $K_1 \cup K_2$ be a link in an $\mathbb{Z}HS^3$ H. Let $(\ell, 1/n)$ be a $1/n$-framed knot in H disjoint from $K_1 \cup K_2$. Then in $H' = \chi(H; (\ell, 1/n))$,

$$\mathrm{lk}_{H'}(K_1', K_2') = \mathrm{lk}_H(K_1, K_2) - n \cdot \mathrm{lk}_H(K_1, \ell) \cdot \mathrm{lk}_H(K_2, \ell),$$

where $K_i' = \chi(K_i; (\ell, 1/n))$.



In particular, when $\mathrm{lk}(K_i, \ell) = 0$, we have

$$\mathrm{lk}_{H'}(\chi(K_1; (\ell, 1/n)), \chi(K_2; (\ell, 1/n))) = \mathrm{lk}_H(K_1, K_2).$$

Definition 1.2. A link $L = K_1 \cup K_2 \cup \cdots \cup K_m$ is called a *boundary link* if K_i bound mutually disjoint compact, connected, oriented surfaces S_i.

For a boundary link, we have $\mathrm{lk}(K_i, K_j) = 0$. Furthermore, by using Lemma 1.1 we have that $\Delta_{K_j \subset H}(t) = \Delta_{\chi(K_j;(K_i,\varepsilon_i)) \subset \chi(H;(K_i,\varepsilon_i))}(t)$. The framing number for K_i is unchanged by surgery along other components.

2. Boundary links and the Casson invariant

In 1985, A. Casson[3] introduced an $\mathbb{Z}$-valued invariant λ for oriented $\mathbb{Z}HS^3$s which satisfies the following properties:

- $\lambda(S^3) = 0$ ($\lambda(H) = 0$ if $\pi_1(H) = 1$)
- $\lambda(-H) = -\lambda(H)$
- $\lambda(H_1 \# H_2) = \lambda(H_1) + \lambda(H_2)$
- $\lambda(\chi(H; (K, 1/n))) - \lambda(H) = \frac{n}{2}\Delta''_{K \subset H}(1)$ [4] [5]

[3]See [1], [16]

[4]This is called the Casson surgery formula. See [6] for the link version.

[5]This is the only restriction in some sense. That is, given two knot-Alexander polynomials $\Lambda_1(t)$ and $\Lambda_2(t)$ with $\varepsilon_1 \Lambda''(1) = \varepsilon_2 \Lambda''(1)$, there are two knots K_1, K_2 in S^3 with $\Delta_{K_i}(t) = \Lambda_i(t)$

Note that $\frac{1}{2}\Delta''_K(1) = a_2(K)$, where a_n denotes the nth coefficient of the Conway polynomial $\nabla_K(z) = \Delta_K(t)|_{t^{-1/2}-t^{1/2}=z}$.

Let H be an $\mathbb{Z}HS^3$, and

$$H = H_1 \overset{(K_1,\varepsilon_1)}{\longrightarrow} H_2 \overset{(K_2,\varepsilon_2)}{\longrightarrow} \cdots \overset{(K_{n-1},\varepsilon_{n-1})}{\longrightarrow} H_n \overset{(K_n,\varepsilon_n)}{\longrightarrow} H'$$

a sequence of surgeries, where K_i is a knot in H_i and $\varepsilon_i \in \{-1,+1\}$. Then there is a boundary link[6] $L = K'_1 \cup \cdots \cup K'_n$ in H such that

$$\chi(H; (K'_1,\varepsilon_1), (K'_2,\varepsilon_2), \cdots, (K'_n,\varepsilon_n)) = H'$$

Then by the Casson surgery formula, we have that

$$\lambda(H') - \lambda(H) = \sum_{i=1}^{n} \varepsilon_i a_2(K'_i).$$

3. Surgery descriptions of knots and cyclic branched covers

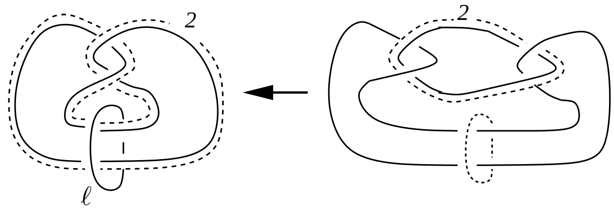
Let $\varphi : M_K^r \to H$ denote the r-fold cyclic covering branched along K. We can compute $|H_1(M_K^r;\mathbb{Z})|$ from the Alexander matrix. Hoste showed the following:

Proposition 3.1 ([6, Theorem 3.2]). Let $\mathcal{D}K$ be the *untwisted doubled knot about* K in H. Then $\lambda(M_{\mathcal{D}K}^r) = \lambda(H) + 2a_2(K)$.

Recall that $\Delta_{\mathcal{D}K}(t) = 1$. Thus, we see that $\lambda(M_K^r)$ cannot be computed from $\Delta_K(t)$ in general.

Let $K \cup \ell$ be a link in H with $\mathrm{lk}(K,\ell) = 0$. Let K^+ denote the preferred longitude for K. Namely $\mathrm{lk}(K,K^+) = 0$. Then in M_ℓ^r, $\varphi^{-1}(K)$ consists of r components. Let $\bar{K}$ be a component of $\varphi^{-1}(K)$ which forms a knot in M_ℓ^r. Let $\bar{K^+}$ be the component of $\varphi^{-1}(K^+)$ corresponding to $\bar{K}$.

Put $\alpha_r(K,\ell) = \mathrm{lk}(\bar{K},\bar{K^+})$. This is an even integer[7]. If $r' >$ (the wrapping number), then $\alpha_{r'}(K,\ell) = $ (const).

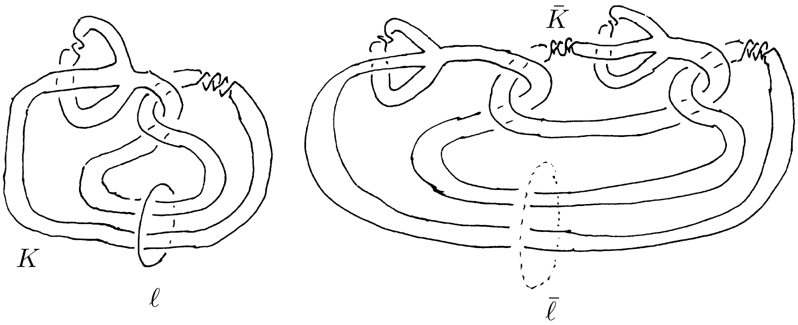


such that $\chi(S^3;(K_1,\varepsilon_1)) = \chi(S^3;(K_2,\varepsilon_2))$ [19, Theorem 1.1]. Can you construct three more knots?

[6]In [11], C. Lescop showed that when $\lambda(H') = \lambda(H)$ there is such a boundary link in H with $\Delta_{K'_i}(t) = 1$. See [8], [19] for more generalization of this result.

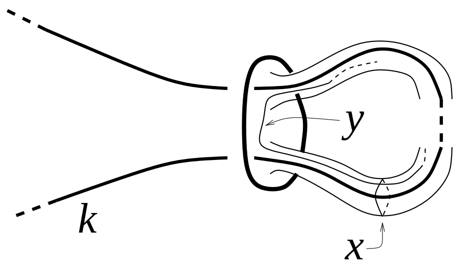
[7]Exercise: Relate $\alpha_r(K,\ell)$ to the Alexander matrix of the knot $\chi(K,(\ell,\pm1))$.

For this Whitehead link $K \cup \ell$, we have $\alpha_r(K, \ell) = 2$ $(r > 1)$. In general, $\Delta_{\bar{K}}(t) \neq \Delta_K(t)$ (even if $\alpha_r(K, \ell) = 0$, even if $K \cup \ell$ is a boundary link.)



$\Delta_K(t) = 1$ (K is a trivial knot) and $\Delta_{\bar{K}}(t) = 1 + 2(t^{1/2} - t^{-1/2})^2$. [8]

Theorem 3.2. Any knot K in an integral homology sphere H has a surgery description $(k; (k_i, \varepsilon_i)_{i=1,2,\ldots,n})$, where k is a trivial knot in S^3 and k_i's are mutually disjoint knots in $E(k)$, with the following properties:



- $\varepsilon_i \in \{-1, +1\}$.
- $k_1, \ldots k_N$ $(N \leq n$ $(N = n$ when $H = S^3))$ bound mutually disjoint genus-one Seifert surfaces S_i's in $E(k)$ on which there are non-separating simple closed curves x_i and y_i such that:
  - x_i intersects y_i transversely in a single point,
  - x_i bounds a disk D_i with $D_i \cap S_i = \partial D_i = x_i$ which meets k in a single point,
  - $\mathrm{lk}(y_i, k) = 0$.
- Each k_j $(N < j \leq n$ if $N \neq n)$ bounds a Seifert surface S_j disjoint from k such that
  - $\mathrm{lk}(k, \ell) = 0$ for each simple closed curve ℓ on S_j,
  - $S_1, S_2, \ldots, S_n$ are mutually disjoint,
- $a_2(K) = -\sum_{i=1}^{N} \varepsilon_i \mathrm{lk}(y_i, y_i^+)$, where y_i^+ denotes the push-forward of y_i with respect to S_i.

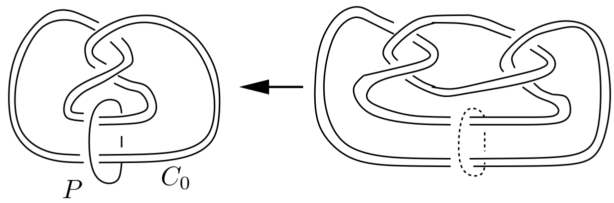
[8]This contradicts the "statement" of [2, Proposition A.7].

- $\lambda(H) = \sum_{i=N+1}^{n} \varepsilon_i a_2(k_i)$.

To illustrate usefulness of such a surgery description, we show the following:

Theorem 3.3. Let K be a satellite knot about a companion C, with the pattern (P, C_0). When the winding number is zero,

$$\lambda(M_K^r) = \lambda(M_P^r) - r\alpha_r(P, C_0)a_2(C).$$



For the surgery description $(k; (k_i, \varepsilon_i)_{i=1,2,\ldots,n})$, in M_P^r the r copies $(\bar{k}_i, \varepsilon_i)_i$ of $(k_i, \varepsilon_i)_i$ give a sequence $M_P^r \to \cdots \to M_K^r$. It is easy to see that $\mathrm{lk}(\bar{x}_i, \overline{x^+}_i) = \alpha_r(P, C_0)$ and $\Delta_{\bar{k}_i}(t) = 1 + \alpha_r(P, C_0)\mathrm{lk}(y_i, y_i^+)(t^{1/2} - t^{-1/2})^2$.

4. Realization problem

Exercise 4.1. For any integer λ, there is a knot K in S^3 such that $\Delta_K(t) = 1$ and $\lambda(M_K^r) = 2r\lambda$ for any natural number $r > 1$. (Hint: Use Hoste's theorem (Proposition 3.1).)

As an application of Theorem 3.3, we have the following:

Proposition 4.2. Let r_0 be a natural number, and λ an integer. Then there is a knot K in S^3 such that $\Delta_K(t) = 1$, $\lambda(M_K^r) = 2r\lambda$ for any natural number $r \neq 1$, r_0, and $\lambda(M_K^{r_0}) = 0$.

Problem 4.3. Let K be a knot in S^3 with $\Delta_K(t) = 1$ (or $a_2(K) = 0$.) Then $\lambda(M_K^r)$ is divided by $2r$. (For unknotting number one knots with $\Delta_K(t) = 1$, this problem is true.)

Exercise 4.4. Let K be a knot in S^3. Then $H_1(M_K^2; \mathbb{Z})$ has no elements of order two. Namely, the order of $\mathrm{Tor}(H_1(M_K^2; \mathbb{Z}))$ is odd.

Exercise 4.5. Given a finitely generated abelian group G without elements of order two, there are knots K in S^3 with $H_1(M_K^2; \mathbb{Z}) = G$.

Exercise 4.6. Study the above propositions for 3-fold cyclic branched covers.
$(H_1(M_K^3; \mathbb{Z}) = G + G$, $|\mathrm{Tor}(G)|$ is odd.)

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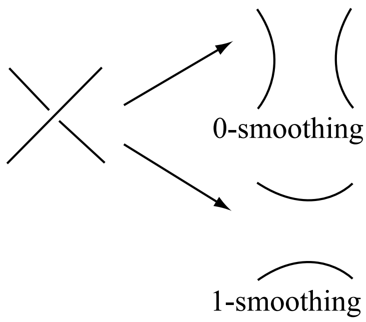
On the Khovanov invariant for links

Norihisa Teshigawara [1]

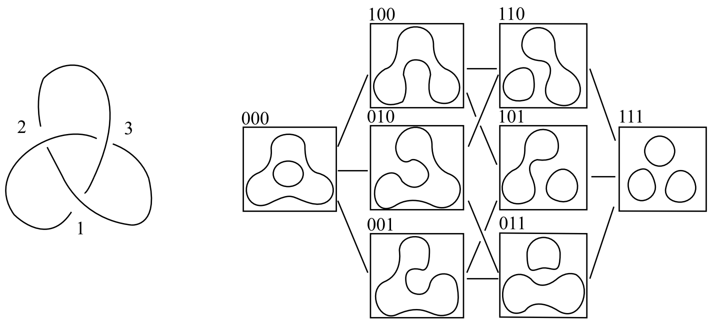
Abstract. In 1999, M. Khovanov constructed an invariant of oriented links. It is a family of (co)homology groups whose graded Euler characteristic is the Jones polynomial. Though his definition is algebraically very complicated, O. Viro simplified it in 2002. In this talk, I will introduce the Khovanov invariant defined by O. Viro and report its related topics.

1. The Khovanov invariant defined by O. Viro

Definition 1. Let D be a diagram of a link in S^3. We exchange a neighborhood of each crossing point of D for either of the following two pictures on the right side. It is called a *Kauffman state* or a *state* of D for short that the disjoint circle(s) given by exchanges like this.



Example 1. The states of the left diagram are like these.



Definition 2. Let s be a state of D. If we assign a plus or minus sign to each circle of s, it is called an *enhanced Kauffman state* or an *enhanced state* of D or s for short.

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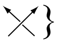
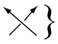
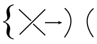
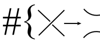
From now on, we assume that links are oriented and crossing points of each diagram are enumerated by natural numbers $1, 2, \cdots, n$.

Definition 3. Define $w(D) := \#\{$$\} - \#\{$$\}$ and $\sigma(s) := \#\{$$\} - \#\{$$\}$. Let $\tau(S)$ be the summation of the signs given to the circles of an enhanced state S. Then,

$$i(S) := \frac{w(D) - \sigma(s)}{2} \qquad j(S) := -\frac{\sigma(s) + 2\tau(S) - 3w(D)}{2}$$

Khovanov chain complices

Definition 4. We call the following three free abelian group a *Khovanov chain group* respectively.

$C(D) := \big\{ \sum_l a_l S_l | a_l \in \mathbb{Z}, S_l : \text{an enhanced state of } D \big\},$
$C^i(D) := \big\{ G \in C(D) | G = \sum_l a_l S_l, i(S_l) = i \big\},$
$C^{i,j}(D) := \big\{ G \in C^i(D) | G = \sum_l a_l S_l, j(S_l) = j \big\}.$

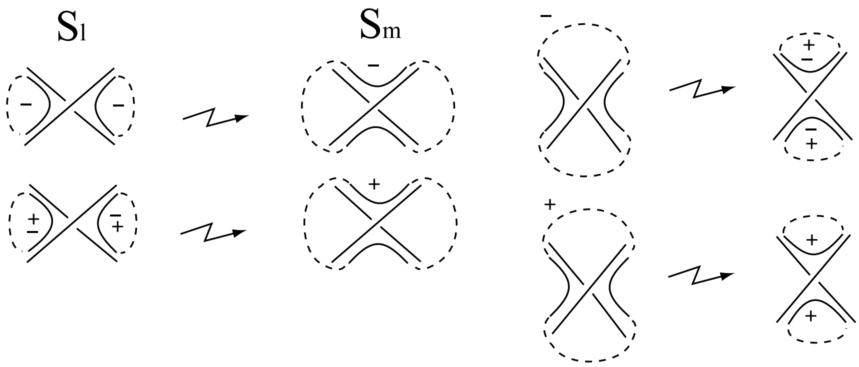
Definition 5. A homomorphism $\partial : C(D) \longrightarrow C(D)$ is defined as below by the incidence numuber (S_l, S_m). $\quad (S_l, S_m \in C(D).)$

$$\partial(\sum_l a_l S_l) = \sum_{l,m} a_l (S_l : S_m) S_m$$

$$(S_l : S_m) = \begin{cases} (-1)^t & \big((S_l, S_m) \text{ satisfys Condition } 1, 2\big) \\ 0 & \big(\text{otherwise}\big) \end{cases}$$

Condition 1. Only at one crossing point of D (Let it have number k), the smoothings of S_l and S_m differ and at this crossing point the smoothing of S_l is 0, while the smoothing of S_m is 1.

Condition 2. The common circles of S_l and S_m have the same signs, and on the signs of the circles of S_l, S_m adjacent to the kth crossing point, they satisfy one of the situations in the next page.



t is the number of 1-smoothings in S_l numerated with numbers greater than k.

Lemma. $\partial^{i+1,j} \circ \partial^{i,j} = 0$.

Notation. $\mathcal{H}^{i,j}(D) := Ker\partial^{i,j}/Im\partial^{i-1,j}$

Definition 6. Let L be an oriented link in S^3 and let D be a diagram of L.
(1) The isomorphism class of $\mathcal{H}^{i,j}(D)$ is called the *Khovanov homology* of L and is denoted by $\mathcal{H}^{i,j}(L)$.
(2)$Kh(L)(t,q) := \sum_{i,j\in\mathbb{Z}} t^i q^j \operatorname{rank}\mathcal{H}^{i,j}(D)$. The polynomial is called the *Khovanov polynomial* of L.

Theorem. $Kh(L)(-1,q) := \hat{J}(L)(q)$.
($\hat{J}(L)(q)$ *is a version of the Jones polynomial such that* $<\emptyset>= 1$.)

2. Related topics

・ **A categorification of the Kauffman bracket skein module**

F を向き付けられた曲面、I を単位区間とする。M. M. Asaeda、J. H. Przytycki、A. S. Sikora 3 氏の共同研究で、O. Viro による Khovanov homology の定義を基に、少なくとも $F \times I$ に関する Kauffman bracket skein module の categorification に成功したようである。

・ **The Khovanov polynomials for the links with trivial Jones polynomials**

絡み目の Jones 多項式が trivial であるとは、自明な絡み目の Jones 多項式と一致することであるとする。M. Thistlethwaite が、2001 年に Jones 多項式が trivial になるような自明でない絡み目の例を 3 つ発表した。それらの Khovanov 多項式を

A. Shumakovitch が作成した KhoHo を用いて計算することが出来た。3 つとも、互いに異なる non-trivial な多項式になる。

・ **A categorification of the HOMFLY polynomial**

P. Ozsváth、Z. Szabó が構成した knot Floer homology は、構成方法は全く異なるのだが Khovanov homology と非常に似た性質を持っている。例えば、あるオイラー標数を取ると Khovanov homology では Jones 多項式と一致するのだが、knot Floer homology では Alexander 多項式と一致する。また交代結び目に関して Khovanov homology(のランク) は Jones 多項式と結び目の符号数で決定できるのに対し、knot Floer homology は Alexander 多項式と結び目の符号数で決定できる。このため、P. Ozsváth、Z. Szabó の論文において「共通の一般化を考えるのは自然なことである。」という指摘がなされていた。

そして、最近 M. Khovanov と L. Rozansky による共著で、それに非常に近いものを構成したという preprint が発表された。具体的には、n が 2 の時に Jones 多項式、n が 0 の時に Alexander 多項式となるように specialize した 1 変数の HOMFLY 多項式に関する categorification を行っている。

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Invariants of 3-manifolds formulated on their presentatinos given by 3-fold branched covering spaces over the 3-sphere

Eri Hatakenaka[1]

Abstract. By a well-known theorem of Hilden and Montesinos, every closed oriented 3-manifold is an irregular 3-fold branched covering of S^3 branched along a knot. Piergallini introduced covering moves, which relates such branch sets representing the same 3-manifold. Hence we can regard the branch set as a presentation of the 3-manifold. We give new invariants of 3-manifolds using this presentation. Then we construct the Dijkgraaf-Witten invariant as an example of the invariants.

1. Introduction

By a well known theorem of Hilden[3] and Montesinos [4], every closed oriented 3-manifold is a 3-fold simple branched covering of S^3 branched along a link. Piergallini [5] introduced the covering moves, which relate two such branch sets representing the same 3-manifold. Hence we can regard the branch set as a presentation of the 3-manifold. In this note we give new invariants of 3-manifolds formulated on the presentations, which are analogous to the quandle cocycle invariant introduced in [1]. Then we construct the Dijkgraaf-Witten invariant as one of the invariants.

2. Covering presentation

For a 3-manifold M a map $p : M \to S^3$ will be a *3-fold branched covering* branched along a link $L \subset S^3$, if

(1) the restriction $p : M - p^{-1}(L) \to S^3 - L$ is a usual 3-fold covering, and
(2) any point $x \in f^{-1}(L)$ has a neighbourhood homeomorphic to $\mathcal{D} \times \mathcal{I}$, where $\mathcal{D}$ is the unit disc in $\mathbb{C}$ and $\mathcal{I}$ is an interval, on which p has the form $p : \mathcal{D} \times I \to \mathcal{D} \times I, (z, t) \mapsto (z^n, t)$, for $n \in \{1, 2, 3\}$.

Such a link L is called the *branch set* of p. To any 3-fold branched covering $p : M \to S^3$, we can assign a homomorphism $\pi_1(S^3 - L) \to \mathfrak{S}_3$, where L is the branch set. A 3-fold branched covering is said to be *simple* if its assigned homomorphism maps each Wirtinger generater to a transposition.

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A *diagram with transpositions* is defined to be a diagram of the branch set of a 3-fold simple branched covering. Note that this diagram is unoriented. Each arc of a diagram with transpositions is associated with a transposition (12), (23) or (31) from its assigned homomorphism. The association is not arbitrary because of the Wirtinger relations at crossings. The transpositions of an over-arc and two under-arcs at a crossing are all the same or all distinct; see Figure 1. In the following figures, the transpositions can be exchanged with each other.

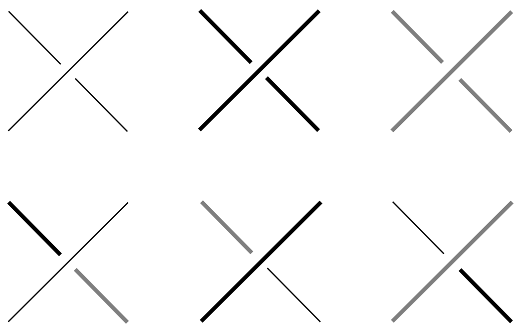


FIGURE 1. Crossings of diagrams with transpositions. The transpositions (12), (23) and (31) are denoted by thin lines, thick lines and gray lines, respectively.

Hilden [3] and Montesinos [4] showed that any closed oriented 3-manifold is homeomorphic to a 3-fold simple branched covering of S^3 branched along a link. Further Piergallini [5] showed that two 3-manifolds are homeomorphic iff their diagrams with transpositions are related by a finite sequence of the 3-move (Figure 2) and the covering moves (Figure 3), up to Reidemeister moves with transpositions. Hence we can regard a diagram with transpositions as a presentation of a 3-manifold. We call it the *covering presentation* of 3-manifolds. Here we have a question;

Make an invariant of 3-manifolds
formulated on their covering presentations.

For the answer of this question, we give a condition of a map which gives an invariant in Section 2, and construct the Dijkgraaf-Witten invariant by giving such a map concretly in Seciton 3.

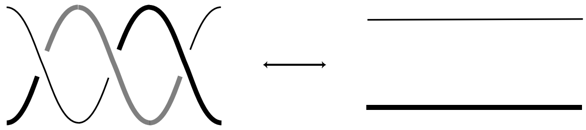


FIGURE 2. The 3-move

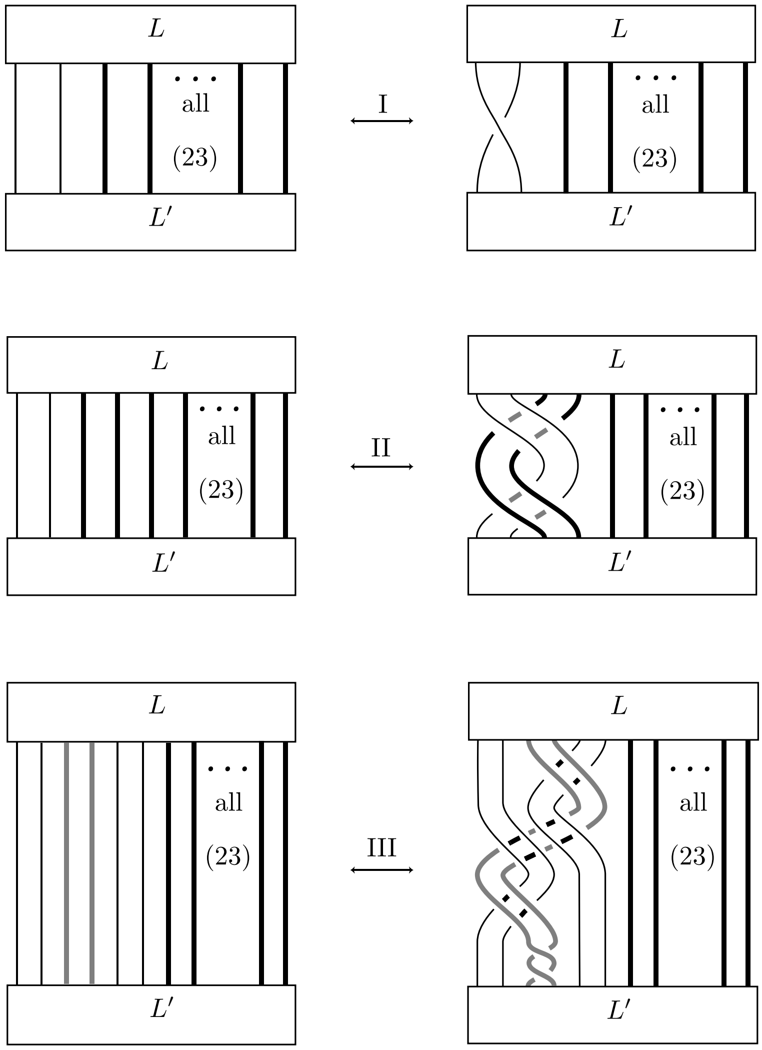


FIGURE 3. The covering moves. All the diagrams consist of a braid joining two arbitrary diagrams with transpositions L and L'.

3. Invariats

For a 3-manifold M, we can compute $\pi_1(M)$ from its covering presentation. Refer to [6] for more details. Using this idea, we define a coloring on a diagram with transpositions which corresponds to a representation from $\pi_1(M)$ to a finite group.

Definition 3.1. Let G be a finite group wtitten multiplicatively, D a diagram with transpositions. A *coloring* on D in G is defined to be a map $\chi : \{\text{arcs of } D\} \to G$, satisfying the following conditions,

(i) at an all the same crossing, $\chi(a) \cdot \chi(b)^{-1} \cdot \chi(c) \cdot \chi(b)^{-1} = 1$, where a and c are the under-arcs and b is the over-arc, as shown in the left hand side in Figure 4.

(ii) At an all distinct crossing, $\chi(a) \cdot \chi(b) \cdot \chi(c) = 1 \in G$, where a is the arc of transposition (12), b is of (23) and c is of (31), as shown in the right hand side in Figure 4. Note that we cannot change the order of the transpositions in the equation.

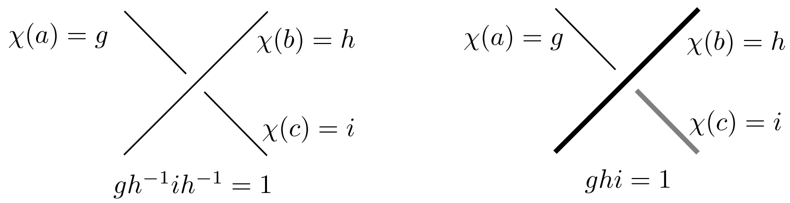


Figure 4. The conditions of colorings on diagrams with transpositions at an all the same crossing (left) and all distinct crossing (right)

Proposition 3.2. The number of colorings on a diagran with transpositions in a finite group is an invariant of the 3-manifold which is given by the diagram.

Sketch of the proof. Given a coloring on a diagram D in a finite group G, we can obtain a representation $\pi_1(M) \to G$, where M is the 3-manifold given by D. Using this representation, we can verify that the number of colorings is preserved under the Reidemeister moves with transpositions, the 3-move and the covering moves. Note that in the covering moves II and III, the colors on the arcs of L' part will be all changed in a rule. □

Proposition 3.3.

$$\sharp\{\text{colors on } D \text{ in } G\} = |G|^2 \cdot \sharp\{\text{representations } \pi_1(M) \to G\}.$$

By Proposition3.2, we construct state sum invariants of 3-manifolds formulated on their covering presentations in the following way.

Let A be a set. If a map

$$X : \{\text{crossings of a colored diagram with transpositions}\} \to A$$

is invariant under the Reidemeister moves, 3-move and covering moves, then the expression

$$\sum_{\chi} \prod_{\tau} X(\tau) \in \mathbb{Z}[A],$$

where the product is taken over all crossings and the sum is taken over all possible colorings in G, is an invariant of 3-manifolds.

4. Dijkgraaf-Witten invariant

We first review the Dijkgraaf-Witten invariant of [2, 7]. Let M be a closed oriented 3-manifold with a triangulation T of N vertices. We give an ordering to the set of the vertices. A *coloring* on T in a finite group G is a map $\omega : \{\text{oriented edges of } T\} \to G$, satisfying the condition depicted in Figure 5, and $\omega(-E) = \omega(E)^{-1}$ for any edge E, where $-E$ is the edge with the opposite orientation.

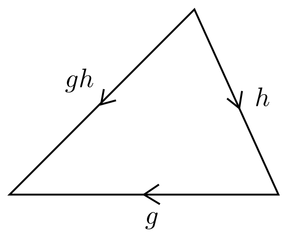


Figure 5. The condition of colorings on triangulations.

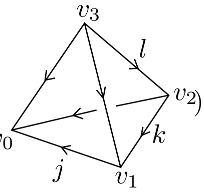
A map $\theta : G \times G \times G \to A$, where A is an abelian group written multiplicatively, is said to be a *3-cocycle* if it satisfies

$$\begin{cases} \theta(1, x, y) = \theta(x, 1, y) = \theta(x, y, 1) = \theta(x, x^{-1}, y) = \theta(x, y, y^{-1}) = 1, \\ \theta(y, z, w) \cdot \theta(xy, z, w) \cdot \theta(x, yz, w) \cdot \theta(x, y, zw) \cdot \theta(x, y, z) = 1, \end{cases}$$

for any $x, y, z, w \in G$. Then the *Dijkgraaf-Witten invariant* is defined by

$$Z_\theta(M) = \frac{1}{|G|^N} \cdot \sum_{\omega} \prod_{\sigma;\ \text{tetrahedron}} W(\sigma) \in \mathbb{Z}[A],$$

where the product is taken over all tetrahedron, the sum is taken over all possible colorings, and

$W($  $) = \theta(j, k, l),$

with $v_0 < v_1 < v_2 < v_3$.

Now let us construct the Dijkgraaf-witten invariant on covering presentations. First we introduce the coloring on the regions of a diagram uniquely given by a coloring on the diagram.

Definition 4.1. Let χ be a coloring on a diagram D in a finite group G. The *coloring on the regions* given by χ is an assignment of an triplet in $G \times G \times G$ to each region of $\mathbb{R}^2 \setminus D$ with the following rules:

(a) The unbounded region is assigned $(1, 1, 1)$.

(b) Two regions separated by an arc are assigned as depicted in Figure 6.

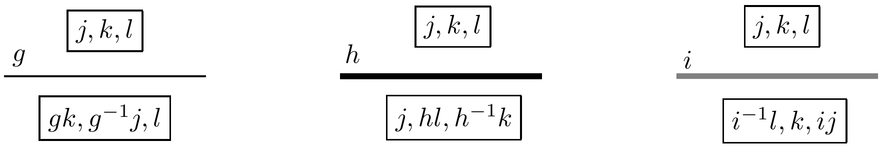
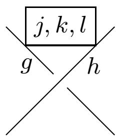


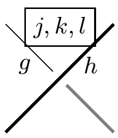
FIGURE 6. The condition of coloring on regions

We can verify that the definition is compatible around any crossing.

Then we define the weight of crossings of a colored diagram with the coloring on regions associated with a 3-cocycle θ in the following equation;

$X($  $) = \theta(g, g^{-1}h, h^{-1}j) \cdot \theta(h, h^{-1}g, k)$

$\cdot\, \theta(hg^{-1}h, h^{-1}g, g^{-1}j) \cdot \theta(h, g^{-1}h, h^{-1}gk),$

$X_\theta($  $) = \theta(g^{-1}, gh, h^{-1}k) \cdot \theta(g^{-1}, j, j^{-1}gh).$

Theorem 4.2. This X_θ satisfies the condition in the box in Section 3. Hence $\sum_\chi \prod_\tau X_\theta(\tau)$ defines an invariant of 3-manifolds.

Proposition 4.3. Let D be a diagram with transpositions, M the 3-manifold given by D. Then

$$\sum_{\chi} \prod_{\tau} X_{\theta}(\tau) = \frac{1}{|G^3|} \cdot Z_{\theta}(M).$$

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A proof of the LMO conjecture

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written by 田所 勇樹 (Yuki Tadokoro)

Abstract. 任意の単連結単純 Lie 群 G に対して、有理ホモロジー 3 球面の LMO 不変量 $\hat{Z}^{\mathrm{LMO}}$ は摂動的不変量 τ^{PG} を再現するという予想（LMO 予想、BGRT or Le の定理）に対して一つの証明を与える。本講演ではコード図が成すグラフ代数上の“微分・積分”について中心的に解説する。これはウェイトシステムと呼ばれる写像を介することで、実際の（方向）微分・（ガウス）積分とみなせる。この性質が証明のキーポイントとなっている。

1. Introduction

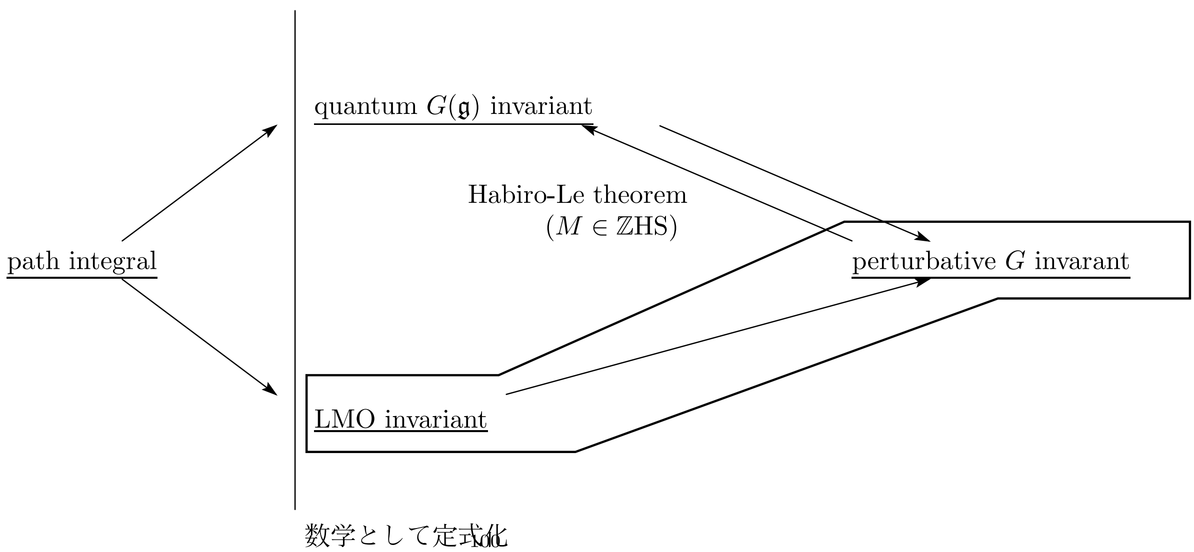
Theorem 1.1. (LMO(Le, J. Murakami, Ohtsuki) conjecture, BGRT(1997) or Le's(?) theorem) G を単連結単純 Lie 群，$\mathfrak{g}$ を G の Lie algebra とする．このとき，

$$\tau^{\mathrm{PG}}(M_K) = |H_1(M_K;\mathbb{Z})|^{-\sharp\Phi_+}\hat{W}_{\mathfrak{g}}(\hat{Z}^{\mathrm{LMO}}(M_K))\big|_{e^h=q}$$

が成立．

今回は，$\mathfrak{g}$： semi-simple Lie algebra.

$$\tau^{\mathrm{PG}}(M) = |H_1(M;\mathbb{Z})|^{-\sharp\Phi_+}\hat{W}_{\mathfrak{g}}(\hat{Z}^{\mathrm{LMO}}(M))|_{e^h=q}, \quad (M \in \mathbb{Q}\mathrm{HS}).$$



$\mathfrak{g} = \mathfrak{sl}_2$ のとき，T. Ohtsuki により解決.

2. Lie algebra の復習

Definition 2.1. $\mathfrak{g}$: finite dimensional Lie algebra /$\mathbb{R}$ or $\mathbb{C}$ とは，$\mathfrak{g}$ が $\mathbb{R}(\mathbb{C})$ 上のベクトル空間で，以下の性質を満たす bilinear form$[\ ,\ ]\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ が存在すること.

$$\begin{cases} [y,x] &= -[x,y] \\ [[x,y],z] &= [x,[y,z]] - [y,[x,z]] \end{cases}$$

ベクトル空間 V に対し，$\mathrm{End}(V)$ は，$[f,g](v) = f \circ g(v) - g \circ f(v)$, $(v \in V,\ f, g \in \mathrm{End}(V))$ と定めることにより，Lie algebra とみなせる．Lie algebra の表現とは，$\mathfrak{g}$ から $\mathrm{End}(V)$ への Lie algebra としての準同型

$$\rho\colon \mathfrak{g} \to \mathrm{End}(V)$$

である．つまり，ρ は，線形写像 $\mathfrak{g} \otimes \mathfrak{g} \to \mathrm{End}(V)$ であり，$[\ ,\ ]$ を

$$\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x) = [\rho(x), \rho(y)], \quad (x, y \in \mathfrak{g})$$

のように保つ.

Example 2.2. (随伴表現)
固定された $x \in \mathfrak{g}$ に対して，

$$\mathrm{ad}(x)\colon \mathfrak{g} \ni y \mapsto [x,y] \in \mathrm{End}(\mathfrak{g})$$

は Lie algebra の表現.

以下，$\mathfrak{g}$ を $\mathbb{C}$ 上のベクトル空間とする．$\mathfrak{g}$ が有限次元なので，$\dim \mathfrak{g} = n$ と考えれば，上記の随伴表現を $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g}) = \mathfrak{g}l(n, \mathbb{C})$ とみなすことができる.

Definition 2.3. (Cartan-Killing form)
symmetric bilinear form

$$B_{\mathfrak{g}}\colon \mathfrak{g} \times \mathfrak{g} \ni (x,y) \mapsto \mathrm{Tr}(\mathrm{ad}(x)\mathrm{ad}(y)) \in \mathbb{C}$$

が $\mathfrak{g}$ の基底の取り方によらず定まる．$B_{\mathfrak{g}}(x,y)$ を (x,y) と記す.

Remark 2.4. $\mathfrak{g}$: semi-simple Lie algebra $\Rightarrow$ $B_{\mathfrak{g}}$: non-degenerate.

semi-simple Lie algebra $\mathfrak{g}$ に対して，Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ が同型を除いて一意的に定まる．$\alpha \in \mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathbb{C})$ に対して，

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g};\ [h, x] = \alpha(h)x\ (h \in \mathfrak{h})\}$$

と定める．

$$\mathfrak{g} = \mathfrak{h} \oplus \Big(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \Big)$$

を root decomposition と呼び，Φ を root と呼ぶ．$\lambda \in \mathfrak{h}$ に対して，

$$\mathrm{ad}(\lambda)(y) = [\lambda, y] = \begin{cases} 0 & (y \in \mathfrak{h}) \\ (\lambda, \alpha)y & (y \in \mathfrak{g}_\alpha) \end{cases}$$

が成り立つ．positive root $\Phi_+ = \{\alpha > 0\}$ に対して，$\rho = \dfrac{1}{2} \sum_{\alpha > 0} \alpha$ とする．

Remark 2.5. $\alpha \in \Phi$ によらず，$\dim \mathfrak{g}_\alpha$ が一定である．

Example 2.6.

$$\mathfrak{sl}_2\mathbb{C} = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \mathbb{C},\ \mathrm{Tr}(A) = a + d = 0 \right\} = \mathbb{C}H \oplus \mathbb{C}E \oplus \mathbb{C}F$$

となる．ただし，

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

であり，$\mathfrak{h} = \mathbb{C}H$ となる．

$\mathfrak{g}$ に関して，$T^m(\mathfrak{g}) = \mathfrak{g}^{\otimes m}$ と定め，テンソル代数を

$$\bigoplus_{m=0}^{\infty} T^m(\mathfrak{g})$$

とする．ただし，$T^0(\mathfrak{g}) = \mathbb{C}$ とする．$\mathfrak{g}$ の元からなる非可換多項式環だと思って良い．$\mathfrak{g}$ のテンソル代数を，$X \otimes Y - Y \otimes X - [X, Y],\ (X, Y \in \mathfrak{g})$ で生成されるイデアルで割った環

$$\mathcal{U}(\mathfrak{g}) = \bigoplus_{m=0}^{\infty} T^m(\mathfrak{g}) / X \otimes Y - Y \otimes X - [X, Y]$$

を univesal enveloping algebra と呼ぶ．

Remark 2.7. (普遍性)
A: associative alegebra を，$[x, y] = xy - yx,\ (x, y \in A)$ と定義して，Lie algebara とみなす．任意の Lie algebra としての準同型 $f \colon \mathfrak{g} \to A$ に対して，以下の図式を可

換とする準同型 $g\colon \mathcal{U}(\mathfrak{g}) \to A$ が同型を除いて一意的に存在する．ただし，$\mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ は自然な埋め込み．

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{f} & A \\ \downarrow & \nearrow_{g} & \\ \mathcal{U}(\mathfrak{g}) & & \end{array}$$

3. Chord diagram

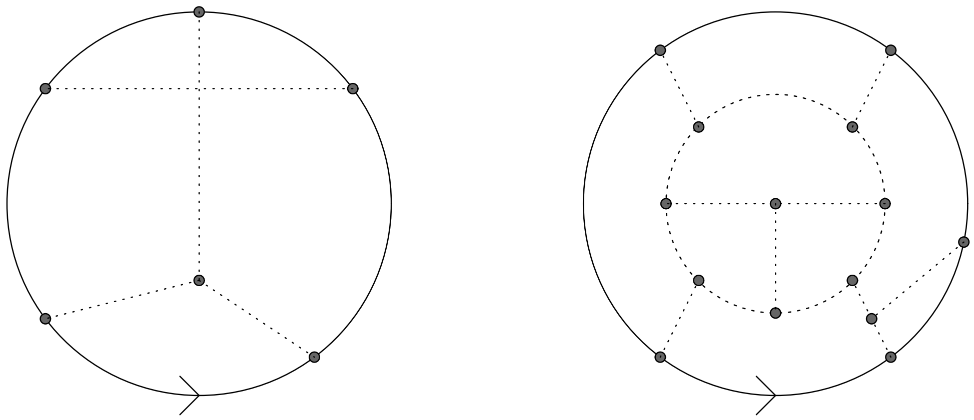


Figure 7. Chord diagram の例

Definition 3.1. Chord diagram を基底に持つ，$\mathbb{Q}$ 上のベクトル空間を AS,IHX,STU の 3 つの relation で割った空間を

$$\mathcal{A}(S^1) = \mathrm{span}_{\mathbb{Q}}\{CD\}/\mathrm{AS}, \mathrm{IHX}, \mathrm{STU}$$

と定める．

台となる S^1 は点線で表し，vertex ● は省略することもある．

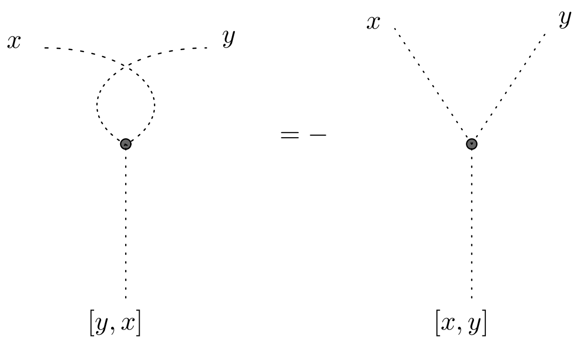
Remark 3.2. $\deg(D) = \dfrac{\sharp\text{vertices}}{2}$ を fix すると，$\mathcal{A}(S^1)$ は有限次元ベクトル空間と思える．

Remark 3.3. STU relation は $\rho = \mathrm{ad}$ とすると，IHX になる．

Definition 3.4.

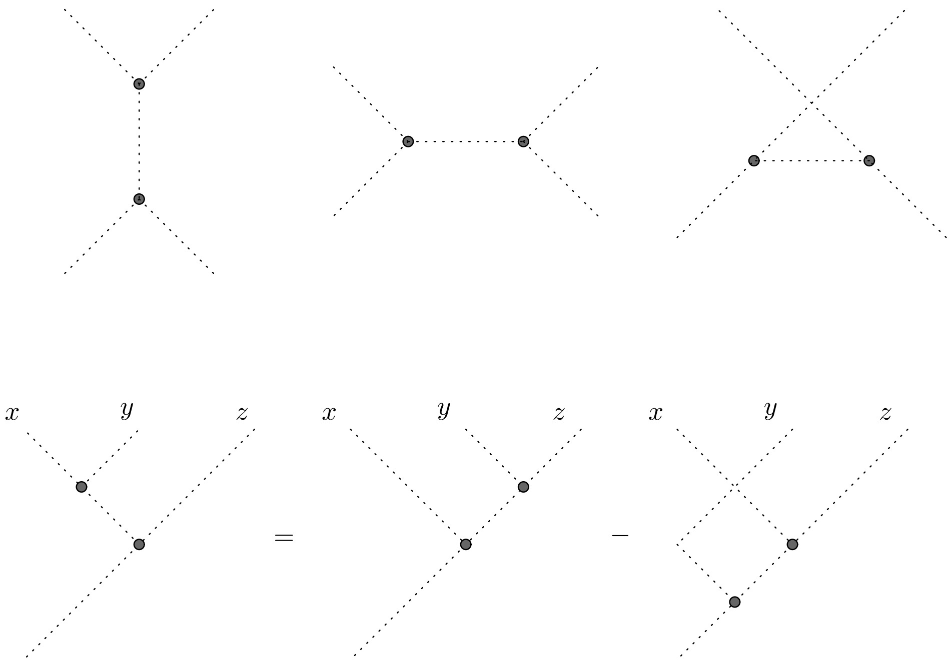
$$\mathcal{A}(\phi) = \mathrm{span}_{\mathbb{Q}}\{\text{3-valent graphs}\}/\mathrm{AS}, \mathrm{IHX}.$$

$$\mathcal{B} = \mathrm{span}_{\mathbb{Q}}\{1, \text{3-valent graphs}\}/\mathrm{AS}, \mathrm{IHX}.$$



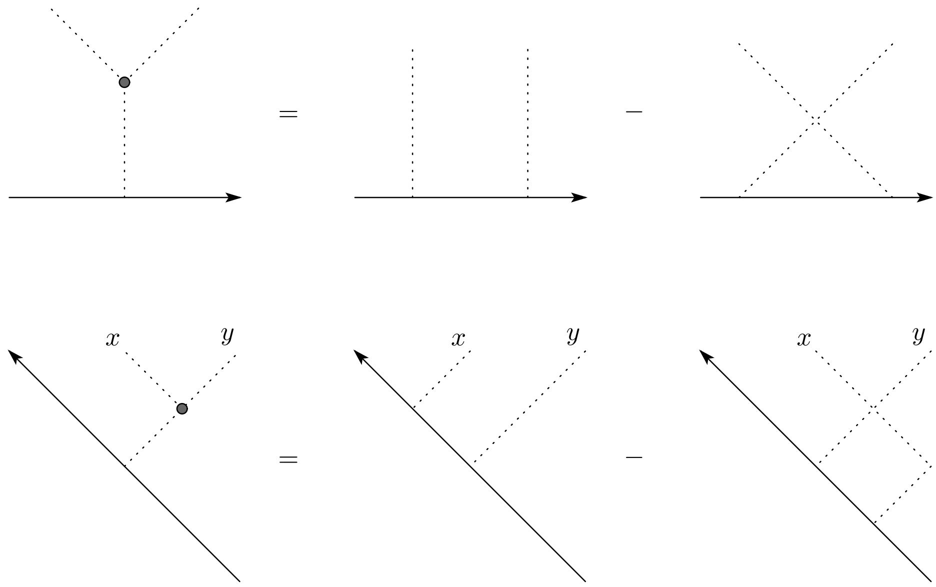
$$[y, x] = -[x, y]$$

FIGURE 8. AS



$$[[x, y], z] = [x, [y, z]] - [y, [x, z]]$$

FIGURE 9. IHX



$\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$

Figure 10. STU

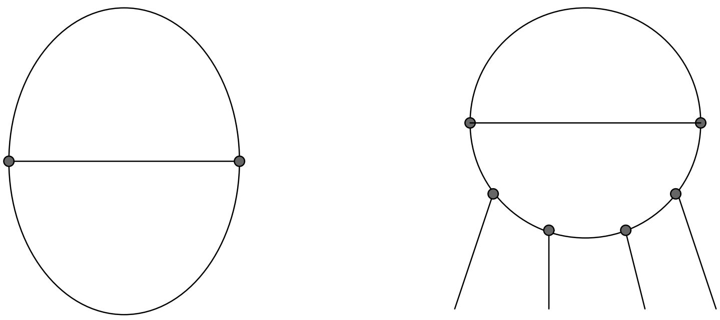


Figure 11. 3-valent graph, $1, 3$-valent graph の例

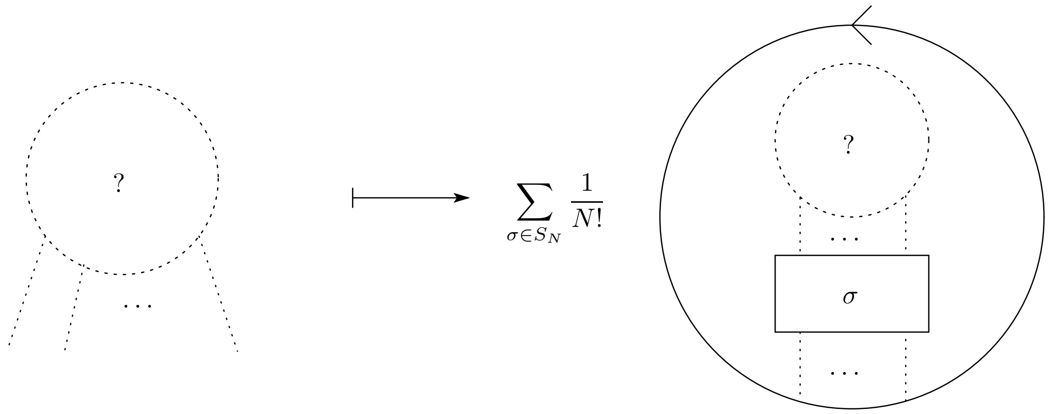


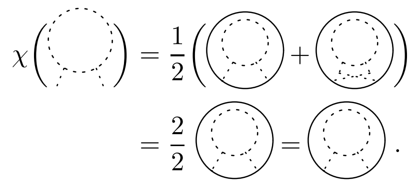
FIGURE 12. $\chi\colon \mathcal{B} \to \mathcal{A}(S^1)$ の定義

The symmetrization map

$$\chi\colon \mathcal{B} \xrightarrow{\cong} \mathcal{A}(S^1)$$

を図 6 のように定める．

Example 3.5. STU relation を用いて，

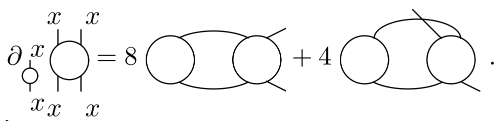


$D, D' \in \mathcal{B}$ に対して，

$$\hat{D} = \partial_D \colon \mathcal{B} \ni D' \mapsto \sum (D, D' \text{を合わせる}) \in \mathcal{B}$$

と定める．“合わせ方”は，legs of $D = m$, legs of $D' = n$ とすると，${}_nP_m = \prod_{k=1}^{m}(n-k+1)$ 通りある．

Example 3.6.



$\frac{d^2}{dx^2}x^4 = 12x^2$ と対応．

Definition 3.7. $D, D' \in \mathcal{B}$ に対して，次が定義できる．

$$\langle D, D' \rangle = \begin{cases} \partial_D D' & (\sharp \text{ legs of } D = \sharp \text{ legs of } D'), \\ 0 & (\text{otherwise}). \end{cases}$$

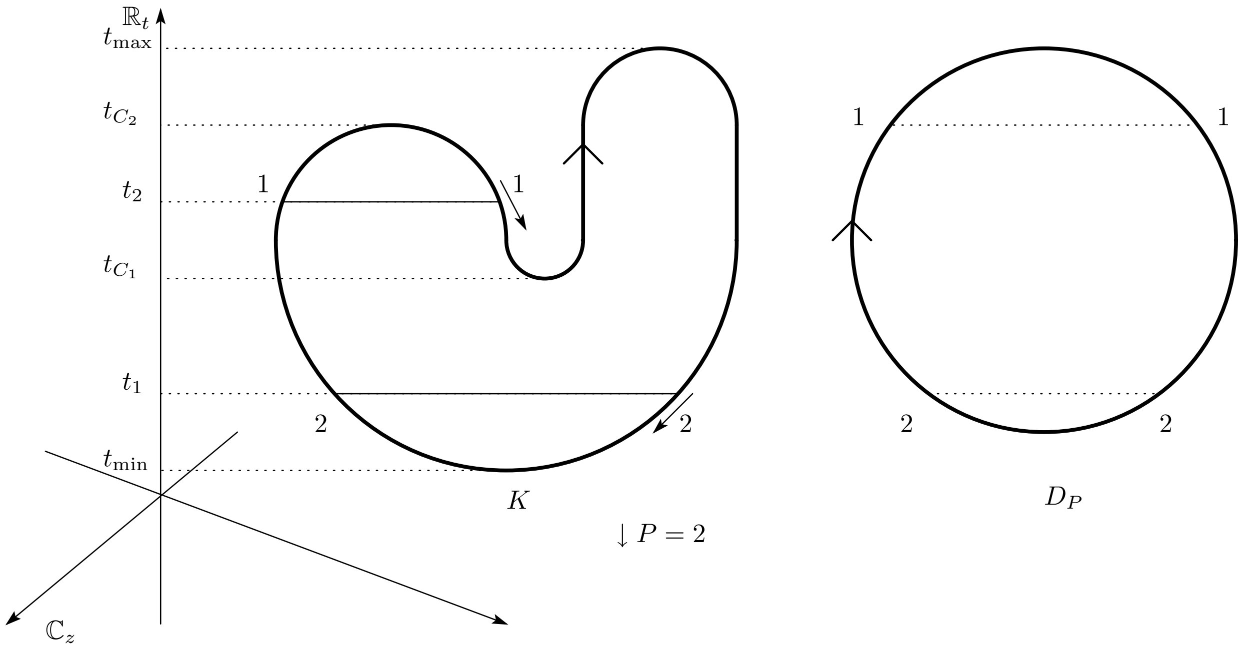


FIGURE 13. K と D_P

t_{C_j} は critical point.

Remark 3.8. $\langle D, D' \rangle$ は $\partial_D D'|_{x=0}$ みたいなもの.

4. Kontsevich integral

Definition 4.1. Morse knot $K \subset \mathbb{R}^3 = \mathbb{R}_t \times \mathbb{C}_z$ の不変量として,

$$\overline{z}(K) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int_{\substack{t_{\min} < t_1 < \cdots < t_m < t_{\max} \\ t_j \text{ non-critical}}} \sum_{\substack{\text{pairings} \\ P = \{(z_j, z_j')\}}} (-1)^{\downarrow P} D_P \bigwedge_{j=1}^{m} \frac{dz_j - dz_j'}{z_j - z_j'}$$

が定まる.

Remark 4.2.

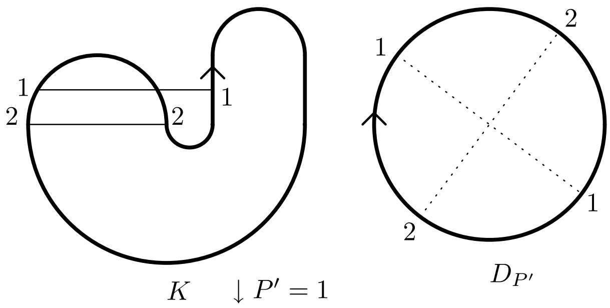


FIGURE 14. $D_{P'}$

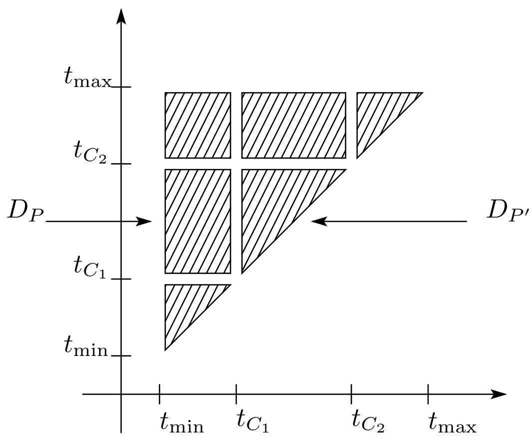
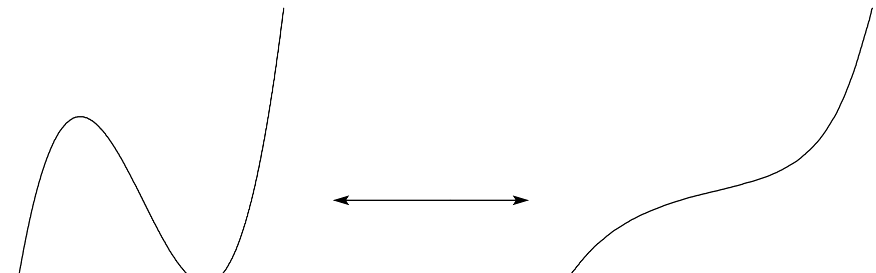
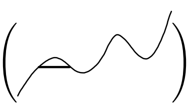


FIGURE 15. 積分範囲

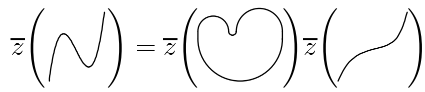


では，$\overline{z}(K)$ は不変にならない．

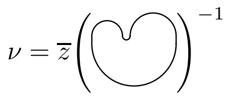
<u>1-term relation</u>

$\overline{z}$ () $= 0$

を chord diagram に入れる．



が成り立つので，



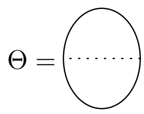
とおいて，

Definition 4.3. (Kontsevich integral)
Morse knot $K \subset \mathbb{R}^3 = \mathbb{R}_t \times \mathbb{C}_z$ が critical point を $c(K)$ 個持つとき，

$$Z(K) = \overline{z}(K)\nu^{\frac{c(K)}{2}} \in \mathcal{A}(S^1)/\text{1-term}$$

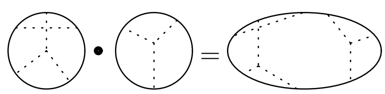
が定まる．

Chord diagram



とおく．

<u>exp について</u>



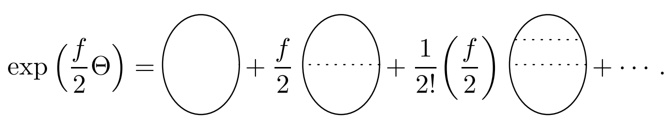
として $\mathcal{A}(S^1)$ に積を入れる．

Definition 4.4. (The integer framed version of Kontsevich integral)
K を integer framed knot, with framing f とするとき，

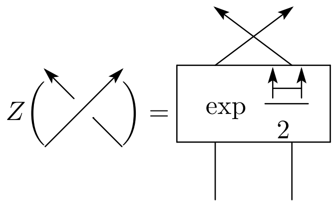
$$\hat{Z}(K) = Z(K)\exp\left(\frac{f}{2}\Theta\right).$$

が定まる．

Remark 4.5.



Example 4.6.



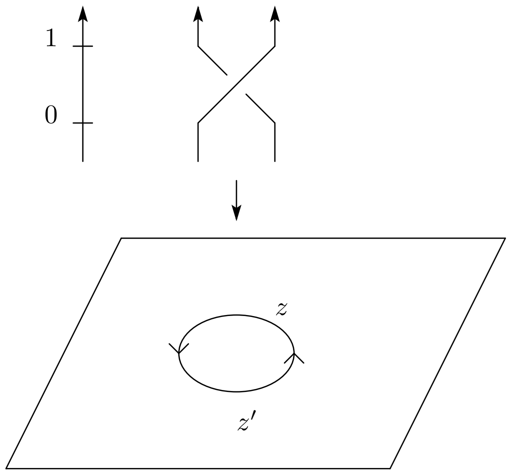
$\because$) $0 < t < 1$ となる t に対して，

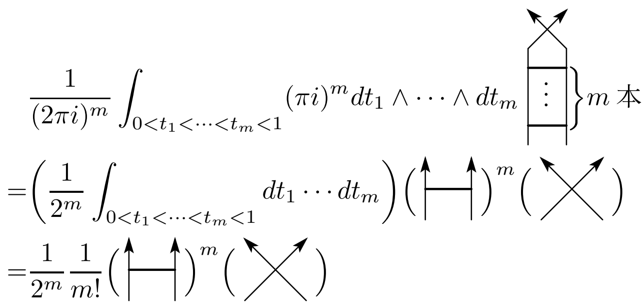
$$z = \exp(\pi i t),\ z' = \exp(\pi i t + \pi i)$$

とおくと，

$$dz = \pi i \exp(\pi i t) dt,\ dz' = \pi i \exp(\pi i t + \pi i) dt$$

が成り立つ．





となるので，結論が言える． $\square$

$$\check{Z}(K) = \nu \hat{Z}(K) \in \mathcal{A}(S^1)$$

は Knot invariant になる．ν は Kirby move で不変になる．

5. LMO invariant

G を単連結単純 Lie 群とし，$\mathfrak{g}$ をその Lie algebra とする．V_λ を，dominant integral weight λ によって prametrize された，$\mathfrak{g}$ の finite dimensional irreducible representation とする．$\mathcal{B}$ に積を，

$$D_1 \cdot D_2 = D_1 \sqcup D_2$$

と導入して，環構造を入れる．また，the formal PBW linear isomorphism を

$$\sigma = \chi^{-1} \colon \mathcal{A}(S^1) \to \mathcal{B}$$

とする．任意の integer framed knot K, with framing f に対して，ある trivalent graph $Y \in \mathcal{B}$ ($\sharp$ 3-valent≥ 1) が存在して，

$$\sigma \check{Z}(K) = \exp\left(\frac{f}{2}\frown\right) \sqcup Y$$

となる．

Definition 5.1. (The formal Gaussian integral of the Kontsevich integral)

$$\int^{\mathrm{FG}} \sigma \check{Z}(K) = \left\langle \exp_{\sqcup}\left(-\frac{1}{2f}\frown\right), Y \right\rangle.$$

Definition 5.2. (LMO invariant) K を framing f の framed knot とする．M_K を K に沿って surgery して得られる rational homology 3-sphere とするとき，

$$\hat{Z}^{\mathrm{LMO}}(M_K) = \frac{\displaystyle\int^{\mathrm{FG}} \sigma \check{Z}(K)}{\displaystyle\int^{\mathrm{FG}} \sigma \check{Z}\left(\bigcirc^{\mathrm{sign}(f)}\right)}.$$

と定める．

Remark 5.3. 逆元は，

$$(1+a)^{-1} = 1 - a + a^2 - a^3 + \cdots$$

で定義する．

Theorem 5.4. (T. T. Q. Le) K を framing f の framed knot とし，M_K を K に沿って surgery して得られる rational homology 3-sphere とする．framing 0 の K と同じ knot を K_0 とおく．このとき，$\tau^{\mathrm{PG}}(M_K)$ は次式で得られる．

$$\frac{1}{|W|}q^{\frac{\mathrm{sign}(f)-f}{2}|\rho|^2}\prod_{\alpha>0}(1-q^{\mathrm{sign}(f)(\rho,\alpha)})\sum_{\substack{\beta\in Y,n\in\mathbb{Z}_{>0}\\ 2|\Phi_+|\le 2j\le n+2|\Phi_+|}} c_{\beta,2j,n}(2j-1)!!\left(-\frac{|\beta|^2}{f}\right)^j h^{n-j}.$$

ただし，$\left(Q_{\mathfrak{g}}(K_0)\big|_{q=e^h}\right)(\lambda-\rho)=\sum_{\substack{\beta\in Y,n\in\mathbb{Z}_{>0}\\ 2|\Phi_+|\le 2j\le n+2|\Phi_+|}} c_{\beta,j,n}\beta^j(\lambda)h^n$，$\beta^j(\lambda)=(\beta,\lambda)^j$ であり，$|W|$ は Weyl 群 W の位数，$\rho=\dfrac{1}{2}\sum_{\alpha>0}\alpha$ である．

Lemma 5.5.
$$\hat{Z}^{\mathrm{LMO}}(M_K)=\langle\Omega,\Omega\rangle\exp\left(\frac{3\mathrm{sign}(f)-f}{48}\theta\right)\times\int^{\mathrm{FG}}\left(\hat{\Omega}^{-1}\sigma\check{Z}(K_0)\right)\exp_{\sqcup}\left(\frac{f}{2}\frown\right).$$

Semi simple Lie algebra $\mathfrak{g}$ に対して，symmetric algebra $S(\mathfrak{g})$ を

$$S(\mathfrak{g})=\bigoplus_{m=0}^{\infty}T^m(\mathfrak{g})/X\otimes Y-Y\otimes X$$

と定め，$S(\mathfrak{g})^{\mathfrak{g}}$ を $\mathfrak{g}$ 作用で不変な，$S(\mathfrak{g})$ 全体とする．The universal weight system は

$$\hat{W}_{\mathfrak{g}}\colon\mathcal{B}\to S(\mathfrak{g})^{\mathfrak{g}}[[h]]$$

と言う写像である．このとき，

$$\hat{W}_{\mathfrak{g}}\circ\sigma\circ\check{z}(K)\colon\mathcal{A}(S^1)\to\mathcal{B}\to S(\mathfrak{g})^{\mathfrak{g}}[[h]]$$

となり，$\lambda\in\mathfrak{h}$ に対して，$J_{\mathfrak{g},V_\lambda}(K)$ は knot の量子不変量で，

$$J_{\mathfrak{g},V_\lambda}(K)=\hat{W}_{\mathfrak{g}}\circ\sigma\circ\check{z}(K)(\lambda)\times\dim V_\lambda$$

が成立する．また，

$$Q_{\mathfrak{g}}(K)(\lambda)=J_{\mathfrak{g},V_\lambda}(K)\times J_{\mathfrak{g},V_\lambda}\left(\bigcirc\right)$$

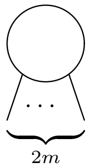
と定める．

r: odd prime としたとき，

$$\tau_r^{\mathrm{PG}}(M)\xrightarrow[\longleftarrow]{\lim}\tau^{\mathrm{PG}}(M)$$

となる．

$\mathcal{B}$ の元 ω_{2m}

$\omega_{2m} =$ 

と定める．また，the modified Bernoulli numbers b_{2m} を以下のベキ級数展開で定義する．

$$\sum_{m=0}^{\infty} b_{2m} x^{2m} = \frac{1}{2} \log\left(\frac{\sinh(x/2)}{x/2}\right).$$

このとき，

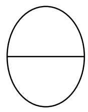
$$\Omega = \exp_{\sqcup} \sum b_{2m}\omega_{2m} \in \mathcal{B}$$

が定まる．

Theorem 5.6. (S. Garoufalidis) $\rho = \dfrac{1}{2}\sum_{\alpha>0}\alpha$ としたとき，

$$\hat{W}_{\mathfrak{g}}(\langle\Omega, \Omega\rangle) = \prod_{\alpha>0} \frac{\sinh \dfrac{(\rho,\alpha)}{2}h}{\dfrac{(\rho,\alpha)}{2}h}$$

が成立．

$\theta =$ 

とおく．

Lemma 5.7.

$$\hat{W}_{\mathfrak{g}}(\theta) = 24(\rho, \rho)$$

が成立．ただし，(,) は Cartan killing form である．

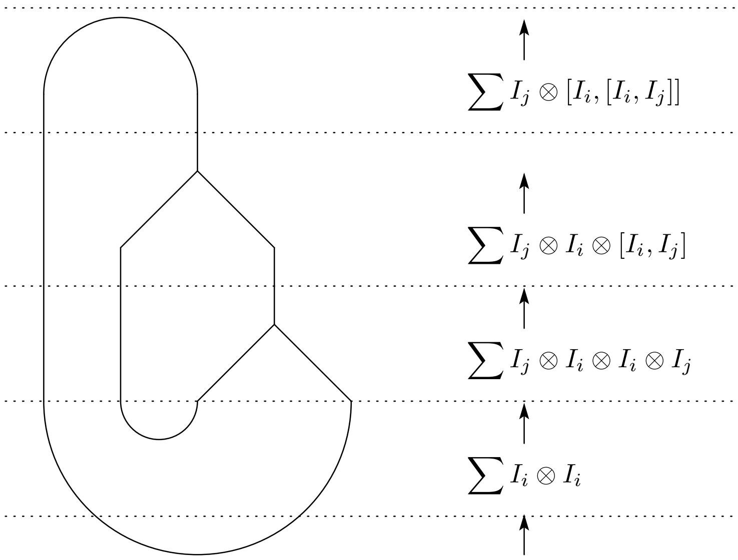
$\hat{W}_{\mathfrak{g}}(\theta)$ を 2 種類の方法で計算する．

<u>その 1(Lie algebra との対応がみやすい)</u>

	$\mathbb{C}$ $\uparrow$ $\mathfrak{g} \otimes \mathfrak{g}$	(X, Y) $\uparrow$ $X \otimes Y$
	$\mathfrak{g} \otimes \mathfrak{g}$ $\uparrow$ $\mathbb{C}$	$\sum_{i=1}^{\dim \mathfrak{g}} I_i \otimes I_i$ $\uparrow$ 1
	$\mathfrak{g}$ $\uparrow$ $\mathfrak{g} \otimes \mathfrak{g}$	$[X, Y]$ $\uparrow$ $X \otimes Y$

ただし，$\{I_\alpha\}$: orthonormal basis with respect to (,).

θ を以下の図のようにみなす.

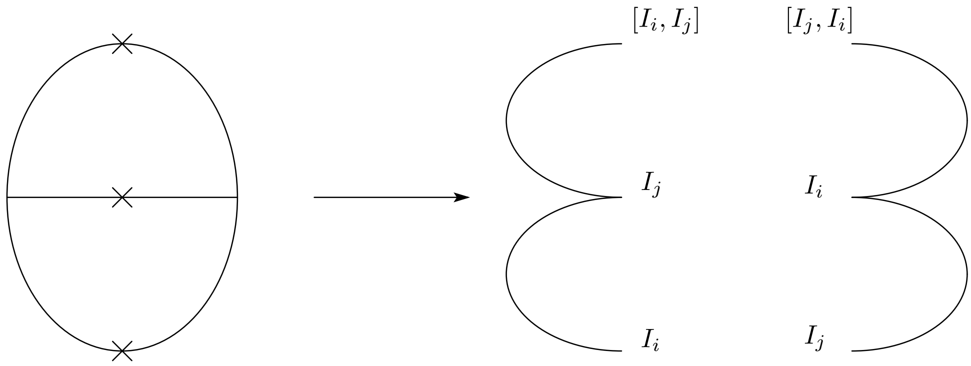


$$\sum (I_j, [I_i, [I_i, I_j]])$$
$$\uparrow$$
$$\sum I_j \otimes [I_i, [I_i, I_j]]$$
$$\uparrow$$
$$\sum I_j \otimes I_i \otimes [I_i, I_j]$$
$$\uparrow$$
$$\sum I_j \otimes I_i \otimes I_i \otimes I_j$$
$$\uparrow$$
$$\sum I_i \otimes I_i$$
$$\uparrow$$
$$1$$

よって，

$$\begin{aligned}
\hat{W}_{\mathfrak{g}}(\theta) &= \sum (I_j, [I_i, [I_i, I_j]]) \\
&= \sum (I_j, \mathrm{ad}(I_i)\mathrm{ad}(I_i) I_j) \\
&= \sum_i \mathrm{ad}(I_i)\mathrm{ad}(I_i) \times \dim \mathfrak{g} \\
&= C_{\mathrm{ad}} \dim \mathfrak{g} \qquad (C_{\mathrm{ad}}\text{はカシミール元}) \\
&= 24(\rho, \rho).
\end{aligned}$$

その2(Lemma 5.7 の証明)



$$\longrightarrow \sum_{i,j}([I_i, I_j], [I_j, I_i]) \times (I_j, I_j) \times (I_i, I_i)$$

$$([x, y], z) = (x, [y, z])$$

が成り立つので，$(I_i, I_j) = \delta_{ij} = \partial_{x_i} x_j$ を用いると，$24(\rho, \rho)$ と $\sum_{i,j}([I_i, I_j], [I_j, I_i]) \times (I_j, I_j) \times (I_i, I_i)$ が一致する． $\square$

Theorem 1.1 の証明

$|H_1(M_K; \mathbb{Z})|^{-\sharp\Phi_+} \hat{W}_{\mathfrak{g}}(\hat{Z}^{\mathrm{LMO}}(M_K))\big|_{e^h=q}$ を計算して，Theorem 5.4 の右辺に一致すれば良い．Lemma 5.5 より，

$$\hat{W}_{\mathfrak{g}}\left(\hat{Z}^{\mathrm{LMO}}(M_K)\right)$$

$$= \underline{\hat{W}_{\mathfrak{g}}\left(\langle \Omega, \Omega \rangle\right) \exp\left(\frac{3\mathrm{sign}(f) - f}{48} \hat{W}_{\mathfrak{g}}(\theta)\right)} \times \hat{W}_{\mathfrak{g}}\left(\int^{\mathrm{FG}} \left(\hat{\Omega}^{-1} \sigma \check{z}(K_0)\right) \exp_{\sqcup}\left(\frac{f}{2} \frown\right)\right)$$

となる．$\underline{\hspace{2cm}}$ 部分は，Theorem 5.6 と Lemma 5.7 により，$e^h = q,\ |\rho|^2 = (\rho, \rho)$ とおけば，

$$
\begin{aligned}
\underline{\hspace{2cm}} &= \prod_{\alpha>0} \frac{\sinh \dfrac{(\rho,\alpha)}{2} h}{\dfrac{(\rho,\alpha)}{2} h} \exp\left(h \frac{3\mathrm{sign}(f) - f}{48} 24(\rho,\rho) \right) \\
&= \prod_{\alpha>0} \frac{\sinh \dfrac{(\rho,\alpha)}{2} h}{\dfrac{(\rho,\alpha)}{2} h} (e^h)^{\frac{3\mathrm{sign}(f)-f}{2}|\rho|^2} \\
&= \boxed{q^{\frac{\mathrm{sign}(f)-f}{2}|\rho|^2} \prod_{\alpha>0} (1 - q^{\mathrm{sign}(f)(\rho,\alpha)})} \times \frac{1}{\prod_{\alpha>0}(\rho,\alpha)} \left(-\frac{\mathrm{sign}(f)}{h} \right)^{\sharp\Phi_+} \quad \cdots \ \diamondsuit
\end{aligned}
$$

である．

一方，$\underline{\hspace{2cm}}$ 以外の部分は，次のように計算される．

Theorem 5.8. (S. Garoufalidis) 以下の図式は可換である．

$$
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{\hat{\Omega}} & \mathcal{B} \\
\hat{W}_{\mathfrak{g}} \downarrow & \circlearrowright & \downarrow \hat{W}_{\mathfrak{g}} \\
S(\mathfrak{g})^{\mathfrak{g}}[[h]] & \xrightarrow[D(j_{\mathfrak{g}}^{\frac{1}{2}})]{} & S(\mathfrak{g})^{\mathfrak{g}}[[h]].
\end{array}
$$

ただし，$D(j_{\mathfrak{g}}^{\frac{1}{2}})$ は Duflo 同型である．

$p \in S(\mathfrak{g})^{\mathfrak{g}}[[h]]$, $\lambda \in \mathfrak{h}$ に対して，$p(\lambda)$ と evaluate でき，

$$
D(j_{\mathfrak{g}}^{\frac{1}{2}}) p(\lambda) = p(\lambda + \rho)
$$

が成立．

この Theorem を用いると，

$$
\begin{aligned}
&\hat{W}_{\mathfrak{g}}\left(\int^{\mathrm{FG}}\left(\hat{\Omega}^{-1}\sigma\check{z}(K_0)\right)\exp_{\sqcup}\left(\frac{f}{2}\frown\right)\right)\\
=&\hat{W}_{\mathfrak{g}}\left(\left\langle\exp_{\sqcup}\left(-\frac{1}{2f}\frown\right),\hat{\Omega}^{-1}\sigma\check{z}(K_0)\right\rangle\right)\\
=&\exp\left(-\frac{h}{2f}\left(\sum_i\partial_{x_i}^2\right)\right)\left(Q_{\mathfrak{g}}(K_0)(\lambda-\rho)\right)\\
=&\exp\left(-\frac{h}{2f}\sum_i\partial_{x_i}^2\right)\left(\frac{\prod_{\alpha>0}(\rho,\alpha)}{\prod_{\alpha>0}(h^{-1}\alpha)^2}\sum c_{\beta,i,n}h^n(h^{-1}\beta)^i\right)\Big|_{x_i=0}\\
=&(-fh)^{\sharp\Phi_+}\frac{\prod_{\alpha>0}(\rho,\alpha)}{\boxed{|W|}}\boxed{\sum c_{\beta,2j,n}\left(-\frac{|\beta|^2}{f}\right)^j(2j-1)!!h^{n-j}}\ \cdots\ \heartsuit
\end{aligned}
$$

となる．

$$
\dim V_{\lambda-\rho}=\prod_{\alpha>0}\frac{(\lambda,\alpha)}{(\rho,\alpha)},\ ((\lambda,\alpha)=\alpha(\lambda))
$$

と

$$
\hat{W}_{\mathfrak{g}}(D)=W_{\mathfrak{g}}(D)\times h^{\deg(D)}
$$

に注意．

あとは，$\diamondsuit,\heartsuit$ と

$$
H_1(M_K;\mathbb{Z})=f\mathrm{sign}(f)
$$

を合わせれば良い． □

Definition 5.9. (The Dedekind symbol)
$\dfrac{p}{q}\in\mathbb{Q}$ に対して，$\mathbb{Q}$ に値を持つ $S\left(\dfrac{p}{q}\right)$ が以下で特徴づけられる．

$$
\begin{aligned}
S(-x) &= -S(x)\\
S(x+1) &= S(x)\\
S\left(\frac{p}{q}\right)+S\left(\frac{q}{p}\right) &= \frac{p}{q}+\frac{q}{p}+\frac{1}{pq}-3\mathrm{sign}(pq).
\end{aligned}
$$

このとき，Lens space $L_{a,b}$ の perturbative invariant が計算できる．

Corollary 5.10.

$$\tau^{\mathrm{PG}}(L_{a,b}) = q^{-\frac{S(\frac{b}{a})}{2}|\rho|^2} \prod_{\alpha>0} \frac{q^{\frac{(\rho,\alpha)}{2a}} - q^{-\frac{(\rho,\alpha)}{2a}}}{q^{\frac{\mathrm{sign}(a)(\rho,\alpha)}{2}} - q^{\frac{-\mathrm{sign}(a)(\rho,\alpha)}{2}}}.$$

Braid indices of surface-knots and colorings by quandles

Kokoro Tanaka[1]

Abstract. The braid index of a surface-knot F is the minimum number among the degrees of all surface braids whose closures are ambient isotopic to F. We give a lower bound of the braid index of a surface-knot using the colorings by a quandle. As an application, we determine the braid indices of S^2-knots for infinitely many examples and give an infinite series of ribbon surface-knots of genus g whose braid indices are $s+2$ for each pair of integers $g \geq 0$ and $s \geq 1$.

1. Surface braids

Please refer to S. Kamada's book [6] for details of surface braid theory.

Definition 1.1. (surface braid)
Let D_1^2 and D_2^2 be 2-disks and X_m a fixed set of m distinct interior points of D_1^2. Let $pr_i : D_1^2 \times D_2^2 \to D_i^2$ be the projection map to the i-th factor for each i $(i = 1, 2)$. A *surface braid of degree m* (or *surface m-braid*) is a compact oriented surface S embedded properly and locally flatly in $D_1^2 \times D_2^2$ such that

(i) the restriction map $pr_2|_S : S \to D_2^2$ is a branched covering map of degree m,
(ii) $\partial S = X_m \times \partial D_2^2 \ \left(\subset D_1^2 \times \partial D_2^2\right)$, and
(iii) the branched covering $pr_2|_S$ is simple, that is, $|S \cap pr_2^{-1}(y)| = m-1$ or m for each $y \in D_2^2$.

Definition 1.2. (equivalence relation)
Two surface braids S and S' are said to be *equivalent* if there is an ambient isotopy $\{h_t\}_{t \in [0,1]}$ such that

(i) $h_0 = \mathrm{id}, h_1(S) = S'$,
(ii) for each $t \in [0,1]$, h_t is fiber-preserving, that is, there is a homeomorphism $\underline{h}_t : D_2^2 \to D_2^2$ such that $pr_2 \circ h_t = \underline{h}_t \circ pr_2$, and
(iii) for each $t \in [0,1]$, $h_t|_{D_1^2 \times \partial D_2^2} = \mathrm{id}$.

Definition 1.3. (closure)
Let S^2 be a 2-sphere obtained from D_2^2 attaching a 2-disk $\overline{D_2^2}$ along the boundary

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of D_2^2. A surface braid S of degree m is extended to a closed surface $\widehat{S}$ in $D_1^2 \times S^2$ $\left(= D_1^2 \times (D_2^2 \cup \overline{D_2^2})\right)$ such that

$$\widehat{S} \cap (D_1^2 \times D_2^2) = S \text{ and } \widehat{S} \cap (D_1^2 \times \overline{D_2^2}) = X_m \times \overline{D_2^2} \ .$$

Identifying $D_1^2 \times S^2$ with the tubular neighborhood of a standard 2-sphere in $\mathbb{R}^4$, we assume that $\widehat{S}$ is a closed oriented surface embedded in $\mathbb{R}^4$. We call it the *closure* of S in $\mathbb{R}^4$.

Theorem 1.4. ([4, 10]) Any oriented surface-link in $\mathbb{R}^4$ is ambient isotopic to the closure of a surface braid of degree m for some m.

Definition 1.5. The *braid index* of a surface-link F, denoted by Braid(F), is the minimum number among the degrees of all surface braids whose closures are ambient isotopic to F.

Remark 1.6. (Known results about braid indices)

- The braid index of the trivial n-component S^2-link is n ($n \geq 1$).
- Braid(F) $= 1 \Longleftrightarrow F$: the trivial S^2-knot.
- Braid(F) $= 2$
  $\Longleftrightarrow F$: the trivial 2-component S^2-knot or the trivial Σ_g-knot ($g \geq 1$).
- Braid(F) $= 3 \Longrightarrow F$: a ribbon surface-link. (it was shown that the other way does not hold.) ([5])
- There are infinitely many ribbon S^2-knots with braid index 3 ([5]).
- There are infinitely many ribbon S^2-knots with braid index 4 ([7]).

2. Quandles and colorings

Definition 2.1. (quandle)
A *quandle* [1, 2] is a set X equipped with a binary operation $(a, b) \mapsto a * b$ such that (i) $a * a = a$ for any $a \in X$, (ii) the map $*a : X \to X$ $(x \mapsto x * a)$ is bijective for each $a \in X$, and (iii) $(a * b) * c = (a * c) * (b * c)$, for any $a, b, c \in X$. A function $f : X \to Y$ between quandles is a *homomorphism* if $f(a * b) = f(a) * f(b)$ for any $a, b \in X$. For each element $a \in X$, the map $*a : X \to X$ is a quandle automorphism of X by (ii) and (iii), and we denote the inverse map $(*a)^{-1}$ by $\overline{*}\, a$.

Definition 2.2. (knot quandle and coloring)
For $n \geq 0$, let M be an oriented $(n+2)$-dimensional manifold and L an oriented n-dimensional manifold embedded in M properly and locally flatly. Let $N(L)$ denote a tubular neighborhood of L in M. Take a fixed point $z \in E(L) = \mathrm{Cl}(M \backslash N(L))$

and let $Q(M, L, z)$ be the set of homotopy classes of paths $\alpha : [0,1] \to E(L)$ such that $\alpha(0) \in \partial E(L)$ and $\alpha(1) = z$. A point $p \in \partial E(L)$ lies on a unique meridional circle of $N(L)$. Let m_p be the loop based at p which goes along this meridional circle in a positive direction. The *knot quandle* of L in M, with the base point z, is a quandle consisting of the set $Q(M, L, z)$ with a binary operation defined by

$$[\alpha] * [\beta] = [\alpha \cdot \beta^{-1} \cdot m_{\beta(0)} \cdot \beta] \ .$$

When $M = \mathbb{R}^{n+2}$, we denote $Q(\mathbb{R}^{n+2}, L, z)$ by $Q(L)$ briefly.

Let F be a surface-link and X a finite quandle. A *coloring* of F by X is a quandle homomorphism $c : Q(F) \to X$ from the knot quandle $Q(F)$ to X. We denote by $\mathrm{Col}_X(F)$ the set of all colorings of F by X. Note that the number of the colorings, $|\mathrm{Col}_X(F)|$, is an invariant of the surface-link F.

3. Main results

Theorem 3.1. Let F be a surface-link which is not a trivial S^2-link. Let X be a finite quandle of order N, where N is a positive integer. If the inequality $|\mathrm{Col}_X(F)| > N^l$ holds for some positive integer l, then we have $\mathrm{Braid}(F) \geq l + 2$.

By using this theorem and Theorem 4.1, we determine the braid indices of S^2-knots for infinitely many examples.

Theorem 3.2. For an odd integer $n \geq 3$, let K_n be the S^2-knot obtained from an $(n, 2)$-torus knot by Artin's spinning construction. Then we have the following.

- (i) The braid index of an S^2-knot $K_n(s)$ is $s+2$, where $K_n(s)$ is the connected sum of s copies of K_n.
- (ii) The braid index of a Σ_g-knot $K_n(s, g)$ is also $s + 2$, where $K_n(s, g)$ is the connected sum of $K_n(s)$ and g copies of a trivial T^2-knot.

Theorem 3.3. For each pair of integers $g \geq 0$ and $s \geq 1$, there exists an infinite series of ribbon surface-knots of genus g whose braid indices are $s + 2$.

4. Lemma, Proposition and Theorem

Theorem 4.1. ([7]) If neither F_1 nor F_2 is a trivial S^2-knot, then the following inequality holds.

$$\mathrm{Braid}(F_1 \# F_2) \leq \mathrm{Braid}(F_1) + \mathrm{Braid}(F_2) - 2$$

Proposition 4.2. Let F be a surface-link which is not a trivial S^2-link. Let α be the minimum number of generators of the knot quandle $Q(F)$. Then we have $\mathrm{Braid}(F) - 1 \geq \alpha$.

Let R_m be a quandle consisting of the set $\{0, 1, \cdots, m-1\}$ with the binary operation defined by $i * j \equiv 2j - i \pmod{m}$, where m is a positive integer. The quandle R_m is called the *dihedral quandle of order* m.

Lemma 4.3. $\left|\mathrm{Col}_{R_m}\big(K_n(s)\big)\right| \leq m^{s+1}$.
The equality sign holds if and only if n is divided by m.

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Surface braid monodromies on a punctured disk

Isao Hasegawa

Abstract. We study surface braid monodromies on a punctured disk by using a monodromy system. It is shown that any monodromy system of a surface braid of degree 3 is ribbon. Here we show that there is a non-ribbon monodromy system.

1. Monodromies on a punctured disk

Let D^2 be a 2-disk and Σ be a set of n interior points in D^2. Fix a base point $y \in \partial D^2$. For a group G, a *G-monodromy* ρ is a homomorphism

$$\rho : \pi_1(D^2 \backslash \Sigma, y) \to G.$$

Two G-monodromies ρ and ρ' are *equivalent*, denoted by $\rho \sim \rho'$, if there exist a homeomorphism $h : (D^2, \Sigma, y) \to (D^2, \Sigma, y)$ and an inner automorphism $\alpha : G \to G$ such that

$$\rho' = \alpha \circ \rho \circ h_*.$$

We often want to classify G-monodromies with some additional conditions under the equivalence $\sim$. For example, Lefschetz fibrations on a sphere with G: the mapping class group on a closed surface. ([2]) or algebraic curves in a projective plane with G: the m-th braid group. ([3]).

To study monodromies easily or systematically, we use a *Hurwitz generating system* $H = (\eta_1, \eta_2, \ldots, \eta_n)$ of $\pi_1(D^2 \backslash \Sigma, y)$ which satisfies the following conditions:

- each η_j surrounds one puncture in a positive direction (see the following figure); and
- $\eta_1 \cdot \eta_2 \cdot \cdots \cdot \eta_n = [\partial D^2]$.

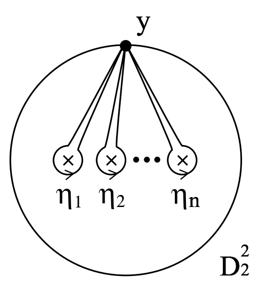

Figure 1

A *G-monodromy system* is an n-tuple

$$\mathrm{MS}(\rho, H) := (\rho(\eta_1), \rho(\eta_2), \dots, \rho(\eta_n)) \in G \times \cdots \times G.$$

A *Hurwitz equivalence* is a equivalence relation on $G \times \cdots \times G$ corresponding to the equivalence of monodromies, which is generated by the following relations:

$$\begin{aligned}(\xi_1, \dots, \xi_j, \xi_{j+1}, \dots, \xi_n) &\sim (\xi_1, \dots, \xi_{j+1}, \xi_{j+1}^{-1}\xi_j\xi_{j+1}, \dots, \xi_n),\\ (\xi_1, \dots, \xi_n) &\sim (\beta^{-1}\xi_1\beta, \dots, \beta^{-1}\xi_n\beta),\end{aligned}$$

where $j = 1, 2, \dots, n-1$ and $\beta \in G$. If monodromy systems MS and MS$'$ are Hurwitz equivalent, we denote it by $\mathrm{MS} \overset{H}{\sim} \mathrm{MS}'$.

Theorem 1.1. Let ρ and ρ' be G-monodromies and H and H' are Hurwitz generating systems. Then

- $\rho \sim \rho'$ if and only if $\mathrm{MS}(\rho, H) \overset{H}{\sim} \mathrm{MS}(\rho', H)$,
- $(\xi_1, \dots, \xi_n) = \mathrm{MS}(\rho, H')$ if and only if $(\xi_1, \dots, \xi_n) \overset{H}{\sim} \mathrm{MS}(\rho, H)$.

2. The surface braid monodromy

Let $D_1^2 \times D_2^2$ be a 4-disk and X_m be a set of m-interior points in D_1^2. A *simple surface braid* of degree m is an oriented compact surface S embedded properly and locally flatly in $D_1^2 \times D_2^2$ which satisfies the following conditions:

- the restriction map $pr_2|_S : S \to D_2^2$ is an m-fold branched covering map,
- $\partial S = X_m \times \partial D_2^2$, and
- $\sharp(S \cap pr_2^{-1}(y)) \geq m-1$ for any $y \in D_2^2$.

We denote a set of branch points in D_2^2 by $\Sigma(S)$. By the condition about ∂S, the number n of elements of $\Sigma(S)$ must be even. For each $y \in D_2^2 \backslash \Sigma(S)$,

$$pr_1(S \cap pr_2^{-1}(y)) \subset \mathrm{Int}(\mathrm{D}_1^2)$$

is a set of distinct m-points in the interior of D_1^2. Thus we get a homomorphism

$$\rho_S : \pi_1(D_2^2 \backslash \Sigma(S), y_0) \to B_m,$$

called a *surface braid monodromy* of S, where B_m is the m-th braid group given as the following way : for each closed curve γ in $D_2^2 \backslash \Sigma(S)$, we define the closed curve $\widetilde{\gamma}$ in the configuration space of unordered m-interior points of D_1^2

$$\widetilde{\gamma}(t) := pr_1(S \cap pr_2^{-1}(\gamma(t))).$$

The fundamental group of this configuration space is isomorphic to B_m. Now we consider monodromy system $\mathrm{MS}(\rho_S, H) = (\rho_S(\eta_1), \rho_S(\eta_2), \ldots, \rho_S(\eta_n)) \in B_m \times \cdots \times B_m$. We say that $\mathrm{MS}(\rho_S, H)$ is *ribbon* if it is equivalent to a system $(\xi_1, \ldots, \xi_n)$ such that $\xi_{2j-1}\xi_{2j} = id_{B_n}$ for each $j = 1, 2, \ldots, \frac{n}{2}$. We say that a surface braid is *ribbon* if its monodromy system is ribbon. Here note that if a surface link has a closed surface braid presentation whose braid is ribbon, then the link is ribbon. However it is not known whether the other way holds or not. We have the following on the ribbonness of a monodromy system.

Theorem 2.1 ([1]). Any monodromy system of a surface braid of degree 3 is ribbon.

Now let φ be a map from B_m to $(\mathbb{Z}_2)^m \rtimes S_m$ given as follows and let π be the projection map from $(\mathbb{Z}_2)^m \rtimes S_m$ to S_m , where S_m is the m-th symmetric group.

$$\varphi(\sigma_i) = \begin{pmatrix} I_{i-1} & & & \\ & 0 & 1 & \\ & t & 0 & \\ & & & I_{m-i-1} \end{pmatrix} = \begin{pmatrix} I_{i-1} & & & \\ & 1 & 0 & \\ & 0 & t & \\ & & & I_{m-i-1} \end{pmatrix} \cdot \begin{pmatrix} I_{i-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_{m-i-1} \end{pmatrix}$$

Observing surface braid monodromies by using homomorphism $s = \pi \cdot \varphi$, we obtain the following theorem. As a corollary of the theorem, we can show that there is a non-ribbon monodromy system.

Theorem 2.2. Let $\mathrm{MS}(\rho_S, H) = (b_1, \cdots, b_n)$ and $\mathrm{MS}(\rho_{S'}, H') = (b'_1, \cdots, b'_n)$ be ribbon monodromy systems. If $(s(b_1), \cdots, s(b_n))$ is Hurwitz equivalent to $(s(b'_1), \cdots, s(b'_n))$, then $(\varphi(b_1), \cdots, \varphi(b_n))$ is Hurwitz equivalent to $(\varphi(b'_1), \cdots, \varphi(b'_n))$.

Corollary 2.3. If $p \equiv 2$ and $q \equiv 2$ (mod4), then a monodromy system $(b_1, b_2, \ldots, b_8)$ is non-ribbon.

$$\begin{aligned} b_1 &= \overline{2}32, & b_5 &= \overline{1}^p\overline{4}^q\overline{2}324^q1^p \\ b_2 &= \overline{3}\,\overline{2}3, & b_6 &= \overline{1}^p\overline{4}^q\overline{3}\,\overline{2}34^q1^p \\ b_3 &= \overline{1}^p\overline{3}231^p, & b_7 &= \overline{4}^q\overline{3}234^q \\ b_4 &= \overline{1}^p\overline{2}\,\overline{3}21^p, & b_8 &= \overline{4}^q\overline{2}\,\overline{3}24^q \end{aligned}$$

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Delta-unknotting numbers and
the Conway polynomials of knots

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Preface

knot の Conway 多項式は, 定数項が 1 で偶数次のみに非零の係数をもつという特徴がある. また, 定数項が 1 で偶数次のみに非零の係数をもつ多項式について, これを Conway 多項式としてもつ unknotting number 1 の knot の構成はよく知られている。さらに, 2 次の係数を Δ-unknotting number とする knot の構成が Hitoshi Murakami により "Delta-unknotting number and the Conway polynomial" (Kobe J. Math. 10 (1993), 17–22) で, 与えられている. こうした結果の拡張として, 定数項が 1 で 2 次の係数が正でそれぞれ異なるような n 個の多項式に対して，それらを Conway 多項式 とし，unknotting number が 1 で Δ-unknotting number が 2 次の係数に一致し, さらに, それらの knot のうちの任意の 2 つについての Δ-Gordian distance が 2 次の係数の差となるような, n 個の knot の構成を与える.

1. Preliminary

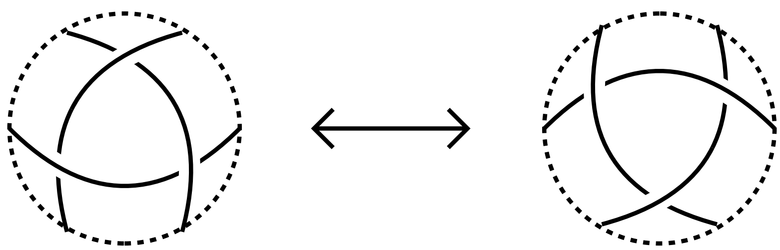
Theorem 1.1 ([3, Louis H. Kauffman])**.** μ components の link L について, 次の 2 条件をみたす.

(1) $\mu - 2 \geq m$ をみたす m について, $a_m(L) = 0$ である.

(2) μ と偶奇を同じくする m について, $a_m(L) = 0$ である.

特に, L が knot のときは, $a_0(L) = 1$ である.

Definition 1.2 ([7, H. Murakami and Y. Nakanishi])**.** 次図で示される knot diagram の local move を Δ-unknotting operation という.



Theorem 1.3 ([7])**.** 任意の knot は, 有限回の Δ-unknotting operation を施すことによって trivial knot になる.

Corollary 1.4. 任意の 2 つの knots K, K' について, K は, 有限回の Δ-unknotting operation を施すことによって, K' になる.

Definition 1.5. knot K の Δ-unknotting number とは, K を trivial knot にするのに必要な Δ-unknotting operation の最小の回数をいう. これを $u^{\Delta}(K)$ と表す.

Definition 1.6. knot K から knot K' への Δ-Gordian distance とは, K を K' にするのに必要な Δ-unknotting operation の最小の回数をいう. これを $d_G^{\Delta}(K, K')$ と表す.

Theorem 1.7 ([9, M. Okada]). 2 つの knot K, K' を $d_G^{\Delta}(K, K') = 1$ となるようなものとするとき,

$$|a_2(K) - a_2(K')| = 1.$$

Corollary 1.8 ([9]). 任意の 2 つの knot K と K' に対して, $d_G^{\Delta}(K, K') - |a_2(K) - a_2(K')|$ は非負の偶数である.

したがって, $u^{\Delta}(K) - |a_2(K)|$ もまた非負の偶数である.

2. Construction of knot for the Conway polynomial

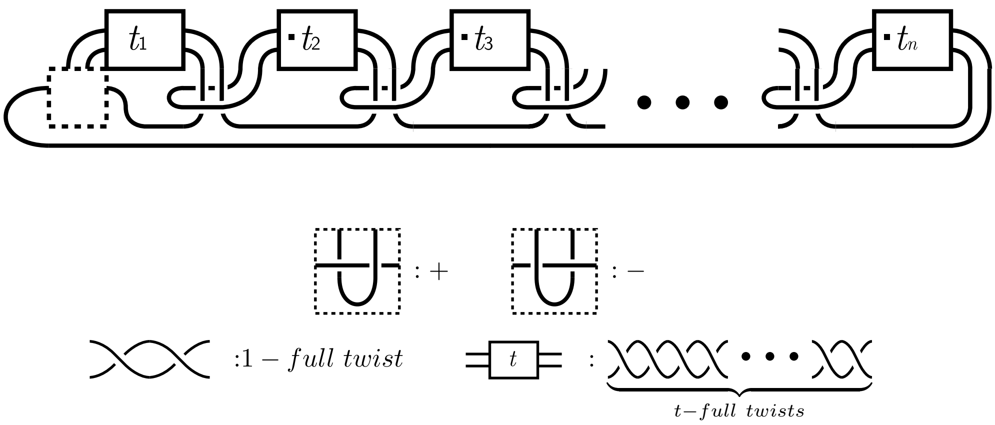
Theorem 1.1 より, knot K の Conway 多項式は

$$1 + a_2 z^2 + a_4 z^4 + \cdots + a_{2m} z^{2m} \quad (m \in \mathbf{N}, a_i \in \mathbf{Z})$$

で表されるが, 逆にこのように表される任意の多項式に対して, その多項式を Conway 多項式としてもつような knot が存在することが示されている.

Theorem 2.1 ([6, H. Murakami]). 任意の多項式
$f(z) = 1 + b_1 z^2 + b_2 z^4 + \cdots + b_n z^{2n}$ $(n \in \mathbf{N}, b_i \in \mathbf{Z})$ に対して, $b_1 \neq 0$ のときは, $u^{\Delta}(K) = |b_1|$ かつ $\nabla_K(z) = f(z)$ となる K が, $b_1 = 0$ のときは, $u^{\Delta}(K) = 2$ かつ $\nabla_K(z) = f(z)$ となる K が存在する.

この証明にあたり, 以下のような knot を考えている.

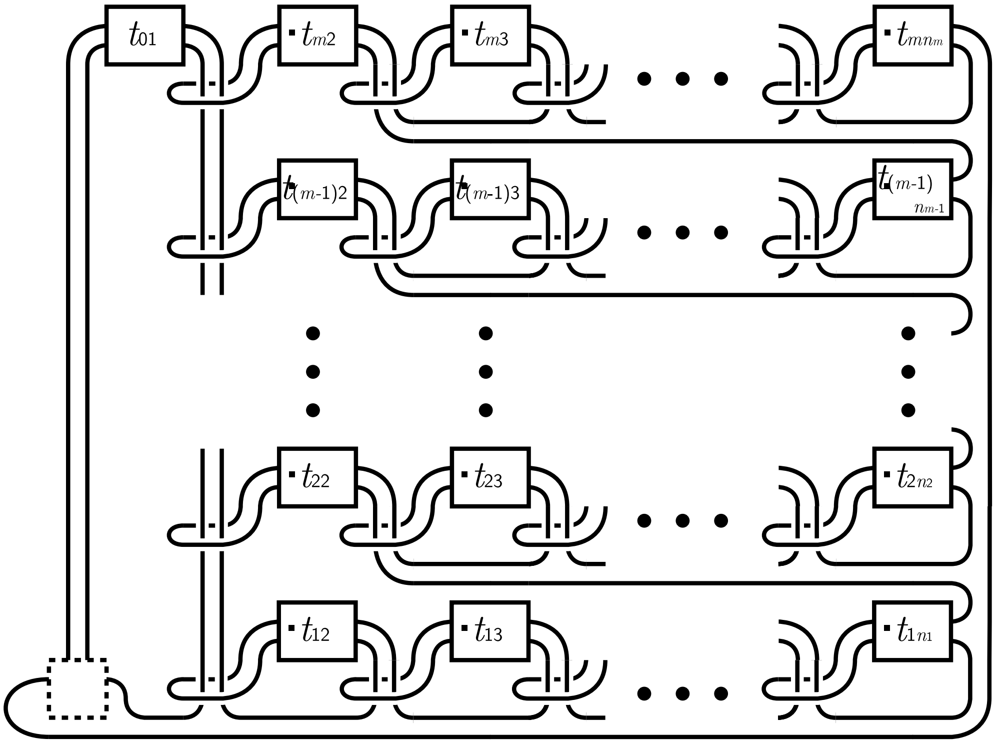


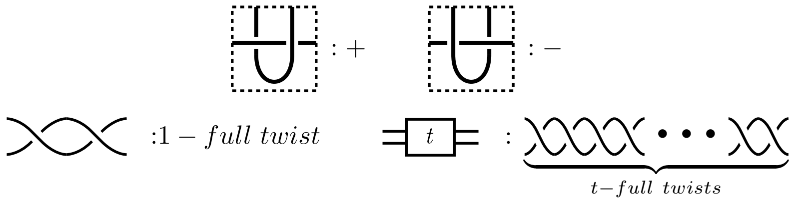
$: K_{\pm}(t_1, t_2, t_3, \cdots, t_n)$

$$\nabla_{K_{\pm}(t_1,t_2,t_3,\cdots,t_n)} = 1 \pm \sum_{i=1}^{n-1} (t_i + 1)(-z^2)^i \pm t_n(-z^2)^n$$

3. Main Theorem

以下のような knot を考える.





この knot を

$$K_{\pm}(t_{01},(t_{12},t_{13},\cdots,t_{1n_1}),(t_{22},t_{23},\cdots,t_{2n_2}),\cdots,(t_{m2},t_{m3},\cdots,t_{mn_m}))$$

と表すことにする.

この knot の Conway 多項式を計算すると,

$$\nabla_{K_{\pm}(t_{01},(t_{12},t_{13},\cdots,t_{1n_1}),(t_{22},t_{23},\cdots,t_{2n_2}),\cdots,(t_{m2},t_{m3},\cdots,t_{mn_m}))}$$
$$= 1 \mp (t_{01}+m)z^2 \pm \sum_{j=1}^{m}\left(\sum_{i=2}^{n_j-1}(t_{ji}+1)(-z^2)^i + t_{jn_j}(-z^2)^{n_j}\right)$$

となる.

この結果から, 次のような Theorem が導き出される.

Theorem 3.1. 任意の m 個の多項式
$\nabla_1(z) = 1 + z^2 + s_{12}z^4 + \cdots + s_{1l_1}z^{2l_1}$,
$\nabla_2(z) = 1 + 2z^2 + s_{22}z^4 + \cdots + s_{2l_1}z^{2l_2}$,
$\vdots$
$\nabla_j(z) = 1 + jz^2 + s_{j2}z^4 + \cdots + s_{jl_j}z^{2l_j}$,
$\vdots$
$\nabla_m(z) = 1 + mz^2 + s_{m2}z^4 + \cdots + s_{ml_m}z^{2l_m} \quad (s_{ji}\in\mathbf{Z})$
に対して, 次の 3 条件をみたす knots $K_1, K_2, \cdots, K_m$ が存在する.

(1) $\nabla_{K_j}(z) = \nabla_j(z)$,
(2) $u^{\Delta}(K_j) = j$,
(3) $d_G^{\Delta}(K_j, K_{j'}) = |j - j'|$.

Proof. $l := \max\{l_1, l_2, \cdots, l_m\}$ とし,
$1\leq^{\forall}j\leq m$ について,
$s_{jk} = 0 \quad (l_j < k\leq l)$
とする.
このとき, 以下の 2 条件を満たす $K_1, K_2, \cdots, K_m$ が一例となる.

(a) $K_1 = K_-(0, ((-1)s_{12} - 1, (-1)^2 s_{13} - 1, \cdots, (-1)^{l-2}s_{1(l-1)} - 1, (-1)^{l-1}s_{1l}))$.
(b) $K_j = K_-(0, (*_1), (*_2), \cdots, (*_j))$
$\Rightarrow K_{j+1} = K_-(0, (*_1), (*_2), \cdots, (*_j),$
$((s_{j2} - s_{(j+1)2}) - 1, -(s_{j3} - s_{(j+1)3}) - 1,$
$\cdots, (-1)^{l-1}(s_{j(l-1)} - s_{(j+1)(l-1)}) - 1, (-1)^l(s_{jl} - s_{(j+1)l})))$.

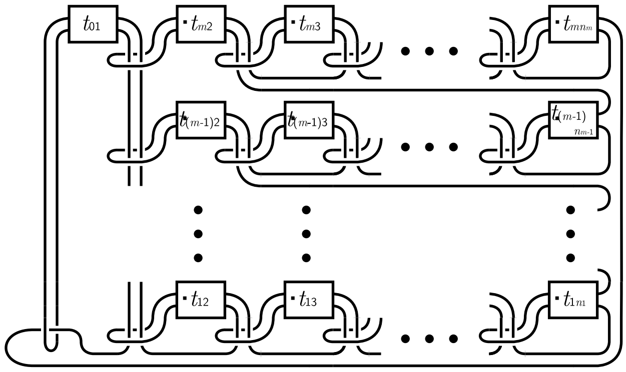
先ほどの結果より,
$t_{01} = 1,\ t_{12} = (-1)s_{12} - 1,\ t_{13} = (-1)^2 s_{13} - 1,$
$\cdots,\ t_{1(l-1)} = (-1)^{l-2}s_{1(l-1)} - 1,\ t_{1l} = (-1)^{l-1}s_{1l}$
$h\geq 1$ について,
$t_{(h+1)2} = (-1)^2(s_{h2} - s_{(h+1)2}) - 1,\ t_{(h+1)3} = (-1)^3(s_{h3} - s_{(h+1)3} - 1),\ \cdots,$
$t_{(h+1)(l-1)} = (-1)^{l-1}(s_{h(l-1)} - s_{(h+1)(l-1)}) - 1,\ t_{(h+1)l} = (-1)^l(s_{hl} - s_{(h+1)l})$ とお

くと，

$$
\begin{aligned}
\nabla_{K_1}(z) &= 1+(t_{01}+1)z^2-\left(\sum_{i=2}^{l-1}(t_{1i}+1)(-z^2)^i+t_{1l}(-z^2)^l\right)\\
&= 1+(0+1)z^2\\
&\quad -\left(\sum_{i=2}^{l-1}(((-1)^{i-1}s_{1i}-1)+1)(-z^2)^i+(-1)^{l-1}s_{1l}(-z^2)^l\right)\\
&= 1+z^2+s_{12}z^4+\cdots+s_{1l}z^{2l}\\
&\quad (s_{1k}=0 \quad (l_1<k\leq l) \quad \text{より，})\\
&= 1+z^2+s_{12}z^4+\cdots+s_{1l_1}z^{2l_1}.
\end{aligned}
$$

$\nabla_{K_j}(z)=1+jz^2+s_{j2}z^4+\cdots+s_{jl_j}z^{2l_j}$ であると仮定する．

$$
\begin{aligned}
\nabla_{K_{j+1}}(z) &= 1+(t_{01}+(j+1))z^2\\
&\quad -\sum_{h=1}^{j+1}\left(\sum_{i=2}^{l-1}(t_{hi}+1)(-z^2)^i+t_{hl}(-z^2)^l\right)\\
&= 1+(t_{01}+j)z^2-\sum_{h=1}^{j}\left(\sum_{i=2}^{l-1}(t_{hi}+1)(-z^2)^i+t_{hl}(-z^2)^l\right)\\
&\quad +z^2-\left(\sum_{i=2}^{l-1}(t_{(j+1)i}+1)(-z^2)^i+t_{(j+1)l}(-z^2)^l\right)\\
&= \nabla_{K_j}(z)+z^2-\sum_{i=2}^{l-1}(t_{(j+1)i}+1)(-z^2)^i-t_{(j+1)l}(-z^2)^l\\
&= 1+jz^2+s_{j2}z^4+\cdots+s_{jl_j}z^{2l_j}\\
&\quad +z^2-\sum_{i=2}^{l-1}((-1)^i(s_{ji}-s_{(j+1)i})-1)+1)(-z^2)^i\\
&\quad -(-1)^l(s_{jl}-s_{(j+1)l})(-z^2)^l\\
&\quad (s_{jk}=0 \quad (l_j<k\leq l) \quad \text{とするので，})\\
&= 1+jz^2+s_{j2}z^4+\cdots+s_{jl}z^{2l}\\
&\quad +z^2-\sum_{i=2}^{l-1}(-1)^i(s_{ji}-s_{(j+1)i})(-z^2)^i\\
&\quad -(-1)^l(s_{jl}-s_{(j+1)l})(-z^2)^l\\
&= 1+(j+1)z^2+s_{(j+1)2}z^4+\cdots+s_{(j+1)l}z^{2l}\\
&\quad (s_{(j+1)k}=0 \quad (l_{j+1}<k\leq l) \quad \text{より，})\\
&= 1+(j+1)z^2+s_{(j+1)2}z^4+\cdots+s_{(j+1)l_{j+1}}z^{2l_{j+1}}.
\end{aligned}
$$



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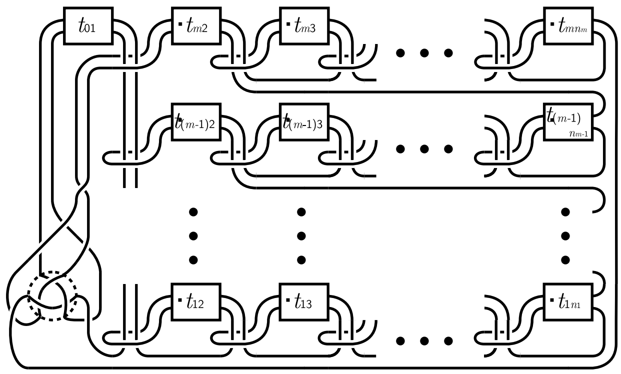
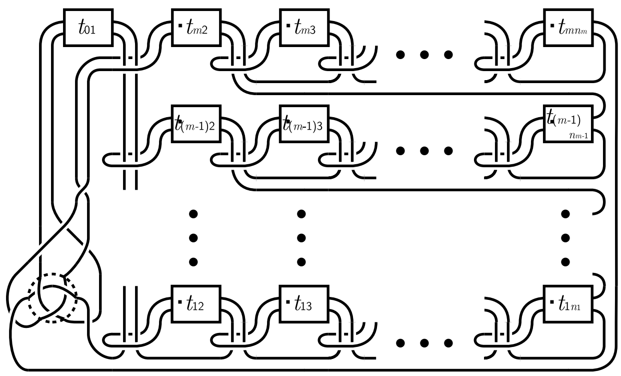


↓

Δ-unknotting operation

↓



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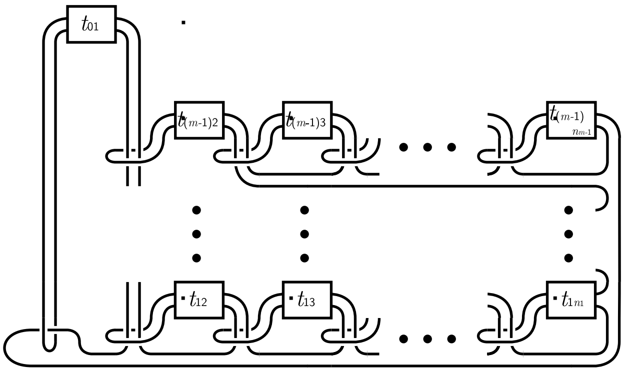


$t_{01} = 0$ のときを考えているので, K_j にこの操作を繰り返していけば, 結局 Δ-unknotting operation を j 回行うことで, trivial knot になる.
よって,

$$u^{\Delta}(K_j) \leq j.$$

また, Corollary 1.8 より, $u^{\Delta}(K_j) - a_2(K_j)$ すなわち $u^{\Delta}(K_j) - j$ は非負の偶数である. したがって,

$$u^{\Delta}(K_j) \geq j.$$

ゆえに,

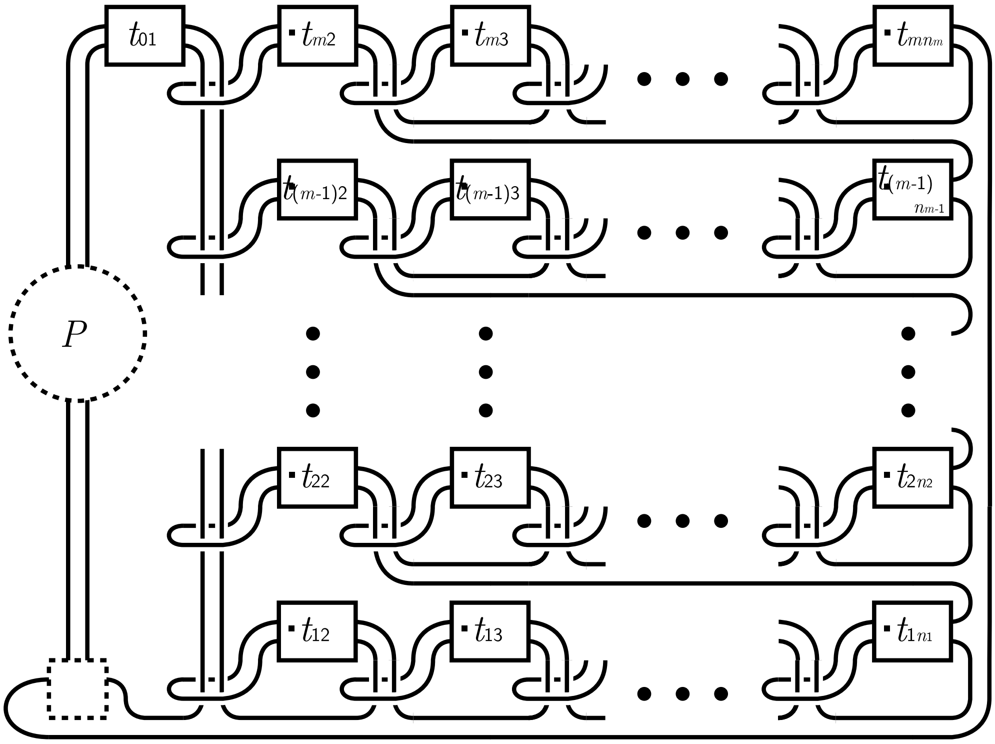
$$u^{\Delta}(K_j) = j.$$

さらに, $d_G^{\Delta}(K_j, K_{j'}) < |j - j'|$ と仮定すると,
$j < j'$ として,
$u^{\Delta}(K_j) + d_G^{\Delta}(K_j, K_{j'}) < j + |j - j'| = j'$.
これは, $u^{\Delta}(K_j) + d_G^{\Delta}(K_j, K_{j'}) \geq u^{\Delta}(K_{j'}) = j'$ であることに矛盾.
また, knot の構成の仕方から $d_G^{\Delta}(K_j, K_{j'}) \leq j' - j = |j - j'|$.
$j > j'$ のときも同様である.
ゆえに,

$$d_G^{\Delta}(K_j, K_{j'}) = |j - j'|.$$

□

次に, 以下のような knot を考える.

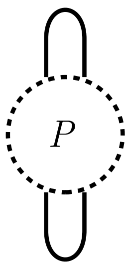


これを,

$$K^P_{\pm}(t_{01},(t_{12},t_{13},\cdots,t_{1n_1}),(t_{22},t_{23},\cdots,t_{2n_2}),\cdots,(t_{m2},t_{m3},\cdots,t_{mn_m}))$$

と表すことにする.

さらに, 以下の knot を P^* と表すことにする.



: P^*

$K_{\pm}(t_{01},(t_{12},t_{13},\cdots,t_{1n_1}),\cdots,(t_{m2},t_{m3},\cdots,t_{mn_m}))$ の計算から以下のことがわかる.

Corollary 3.2. $\nabla_{P^*}(z) = 1$ ならば,

$$\nabla_{K^P_\pm(t_{01},(t_{12},t_{13},\cdots,t_{1n_1}),(t_{22},t_{23},\cdots,t_{2n_2}),\cdots,(t_{m2},t_{m3},\cdots,t_{mn_m}))}$$
$$= 1 \mp (t_{01}+m)z^2 \pm \sum_{j=1}^{m}\left(\sum_{i=2}^{n_j-1}(t_{ji}+1)(-z^2)^i + t_{jn_j}(-z^2)^{n_j}\right).$$

すなわち,

$$\nabla_{K^P_\pm(t_{01},(t_{12},t_{13},\cdots,t_{1n_1}),(t_{22},t_{23},\cdots,t_{2n_2}),\cdots,(t_{m2},t_{m3},\cdots,t_{mn_m}))}$$
$$= \nabla_{K_\pm(t_{01},(t_{12},t_{13},\cdots,t_{1n_1}),(t_{22},t_{23},\cdots,t_{2n_2}),\cdots,(t_{m2},t_{m3},\cdots,t_{mn_m}))}.$$

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Delta link-homotopy on spatial graphs

Ryo Nikkuni[1]

Abstract. Delta link-homotopy is an equivalence relation on oriented links generated by delta moves on the same component and ambient isotopies, and extended to spatial graphs naturally. In this talk, we will explain the content of the papers [18, 19, 20]. We also refer the reader to [22, 23, 24] for their outlines in Japanese.

In this note we will discuss about

- the relation between delta link-homotopy and the other equivalence relations and
- (complete) classifications of spatial embeddings of certain graphs up to delta link-homotopy.

Acknowledgment.

This lecture note is based on the notes made by Kazuhiro Ichihara. All of the figures in this note were illustrated by Reiko Shinjo. The author wishes to express his deep gratitude to them.

0. Equivalence relations on spatial graphs

Throughout this talk, we only consider a finite graph without free vertices (i.e., vertices with valency 0, 1). We always regard a graph G as a 1-dimensional CW- or simplicial complex as usual. We call an embedding $f : G \to S^3$ a *spatial embedding of* G, or simply, a *spatial graph*.

Example 0.1. Throughout the following, all vertices and edges of graphs will be assumed to be labeled with numbers.

In this section we recall basic definitions, which will be used throughout the talk. First we give a summary of known equivalence relations for spatial graphs.

Definition 0.2 (equivalence relations, [32])**.** Let f, g be spatial embeddings of a graph G.

(1) f and g are *ambient isotopic* if there exists an orientation preserving homeomorphism $\Phi : S^3 \to S^3$ such that $f \circ \Phi = g$ holds.

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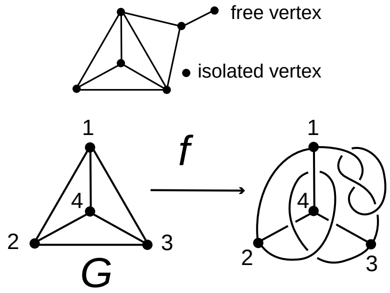


Fig. 0.1

It is known by [6, 35] that this is equivalent to that f and g are transformed into each other by the *Reidemeister moves*; that are (I), (II), (III) (original) Reidemeister moves for knots, and (IV), (V), see Fig. 0.2.

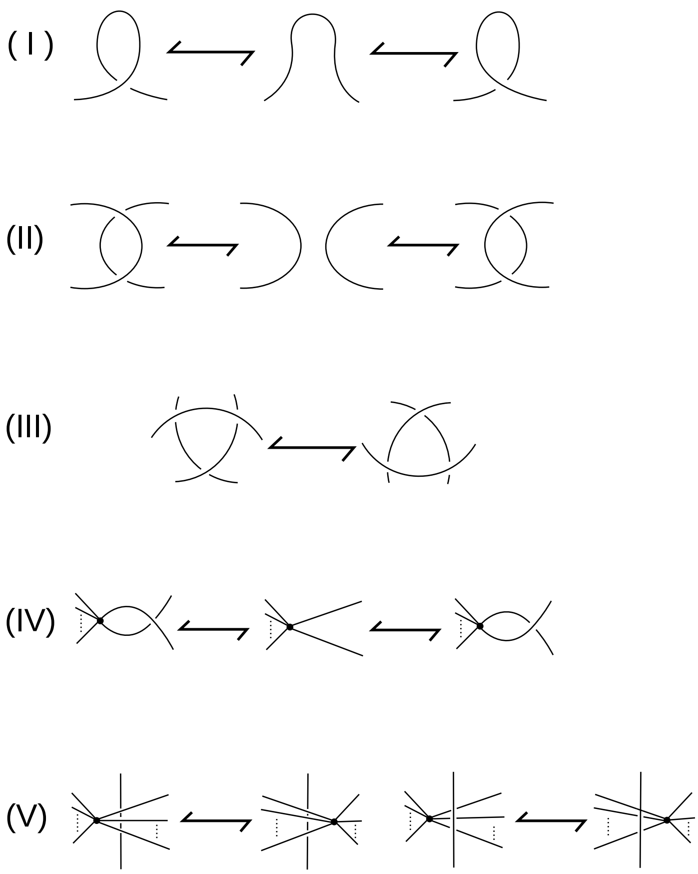


Fig. 0.2. Reidemeister moves

This is also equivalent to that there exists a level preserving locally flat embedding $\Phi : G \times I \to S^3 \times I$ between f and g. Here $\Phi : G \times I \to S^3 \times I$ is said to be

(a) *between* f *and* g if there is a real number $\varepsilon > 0$ such that $\Phi(x,t) = (f(x),t)$ for any $x \in G$, $0 \le t \le \varepsilon$ and $\Phi(x,t) = (g(x),t)$ for any $x \in G$, $1-\varepsilon \le t \le 1$,

(b) *locally flat* if every point $p \in \Phi(G \times I)$ has a neighborhood N such that $(N, N \cap \Phi(G \times I))$ is pairwise homeomorphic to the standard disk pair (D^4, D^2) or $(D^3, X_n) \times I$, where (D^3, X_n) denotes the pair as illustrated in Fig. 0.3 and

(c) *level preserving* if there is a map $\phi_t : G \to S^3$ for each $t \in I$ such that $\Phi(x,t) = (\phi_t(x),t)$ for any $x \in G$, $t \in I$.

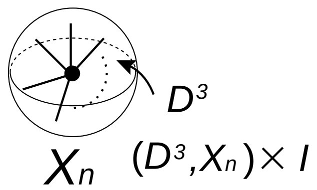


Fig. 0.3

(2) f and g are *cobordant*, denoted by $f \overset{\text{(Cob)}}{\sim} g$, if there exists a locally flat embedding $\Phi : G \times I \to S^3 \times I$ between f and g (see Fig. 0.4).

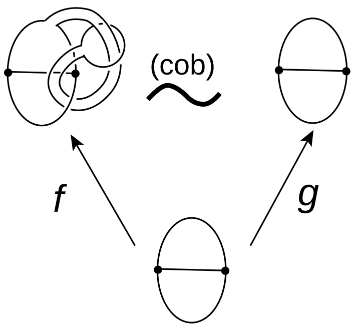


Fig. 0.4. Cobordant spatial embeddings

(3) f and g are *isotopic*, denoted by $f \overset{\text{(Iso)}}{\sim} g$, if there exists a level preserving embedding $\Phi : G \times I \to S^3 \times I$ between f and g. Isotopy on spatial graphs is an application of Alexander trick, see Fig. 0.5.

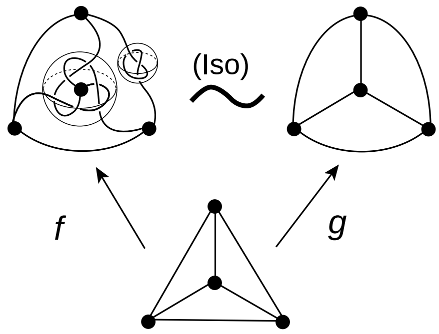


Fig. 0.5. Isotopic spatial embeddings

(4) f and g are *I-equivalent* if they are "cobordant" or "isotopic". Precisely they are I-equivalent if there exists an embedding $\Phi : G \times I \to S^3 \times I$ between f and g.

(5) f and g are *edge-homotopic*, denoted by $f \overset{\text{(EH)}}{\sim} g$, if f and g are transformed into each other by self crossing changes and ambient isotopic, see Fig. 0.6.

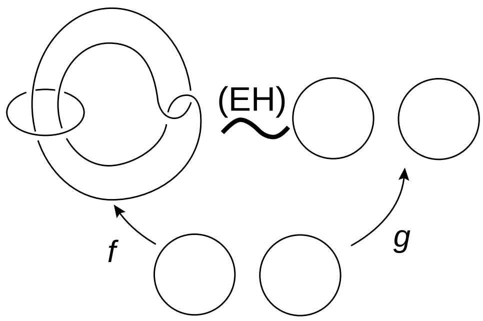


Fig. 0.6. Edge-homotopic spatial embeddings (2-component links)

Here *self crossing changes* mean the crossing changes on the same spatial edge as illustrated in Fig. 0.7.

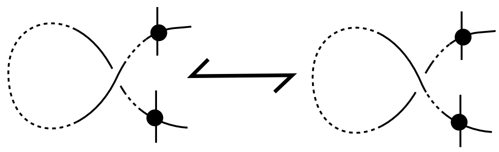


Fig. 0.7. Self crossing change

Remark. Edge-homotopy on spatial graphs is a generalization of link-homotopy on links in the sense of Milnor [10].

A graph G is said to be *planar* if there exists an embedding $f : G \to S^2$, and is said to be *non-planer* otherwise. By the well-known Kuratowski's theorem, G is non-planar if and only if it contains a subgraph which is homeomorphic to K_5 or $K_{3,3}$ as illustrated in Fig. 0.8

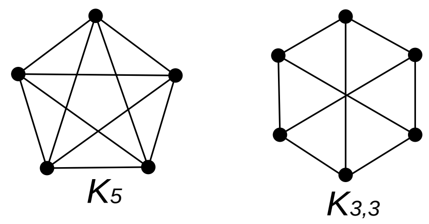


Fig. 0.8

Definition 0.3. Suppose that G is planar. Let f be a spatial embedding of G.

(1) f is *trivial* if f is ambient isotopic to an embedding $h : G \to S^2 \subset S^3$.

(2) f is *slice* if f is cobordant to the trivial embedding.

Remark. The well-definedness of the trivial embedding was shown in [8]: Any two embeddings of G into $S^2 \subset S^3$ are actually ambient isotopic.

1. Delta edge-homotopy, Delta vertex-homotopy

In this section, we introduce two more equivalence relations, which can be regarded as a natural extension of delta link-homotopy on links. These are the main subjects in this talk.

A *delta move* is a local move on links as illustrated in Fig. 1.1. This move was introduced in [9, 12] and showed that it is an unknotting operation for knots. This is naturally extended to a local move to spatial graphs.

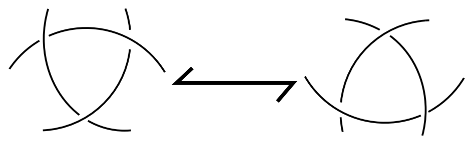


Fig. 1.1. Delta move

Remark. The delta move is not an unlinking operation for links. Because it keeps linking numbers of links invariant.

Adding certain restrictions, let us introduce two similar local moves. A *self delta move* is defined to be the delta move on the same spatial edge as illustrated in Fig. 1.2.

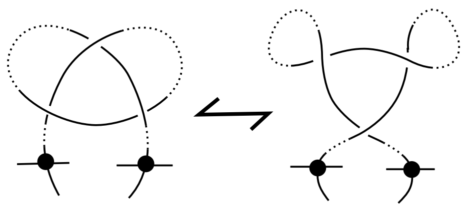


Fig. 1.2. Self delta move

A *quasi adjacent-delta move* is defined to be the delta move on exactly two adjacent spatial edges as illustrated in Fig. 1.3.

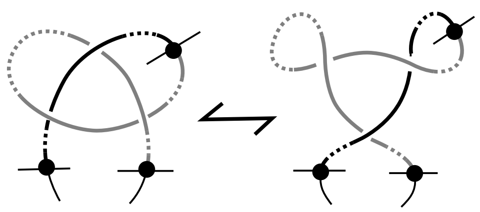


Fig. 1.3. Quasi adjacent-delta move

By using these moves, two equivalence relations are defined.

Definition 1.1. ([18]) Two spatial embedding $f, g : G \to S^3$ are

(1) *delta edge-homotopic*, denoted by $f \overset{(\mathrm{DEH})}{\sim} g$, if f and g are transformed into each other by self delta moves and ambient isotopies,

(2) *delta vertex-homotopic*, denoted by $f \overset{(\mathrm{DVH})}{\sim} g$, if f and g are transformed into each other by quasi adjacent-delta moves and ambient isotopies,

Remark.

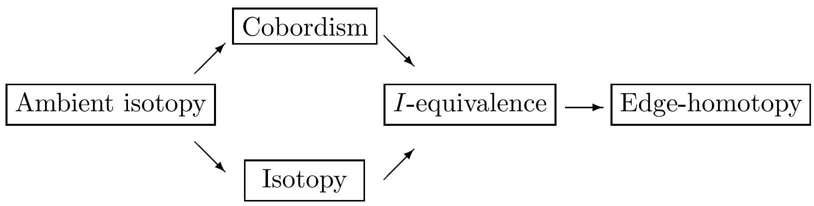
- In the case of $G = S^1 \amalg \cdots \amalg S^1$, delta edge-homotopy and delta vertex-homotopy are equivalent as equivalence relations, for they are natural extension of *delta link-homotopy* on links [29, 30, 17, 13, 14, 15].

- In general, a self delta move is not an unlinking operation. However it actually is for ribbon links and for 2-component boundary links [29, 30]. It is still open for boundary links with three or more components.

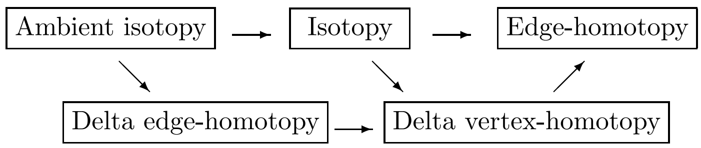
A motivation to study these moves, at least for the author, is to classify spatial graphs without considering ‘local knots’. In some sense, the properties of spatial graphs invariant under self crossing changes or self delta moves must be essential in spatial graph theory independent from knot theory.

2. Relation to the other equivalence relations

Theorem 2.1. The following implications hold [32].



Moreover the following implications hold [18].



In the theorems above, (1) → (2) means that $f \overset{(1)}{\sim} g$ implies $f \overset{(2)}{\sim} g$. It is remarked that there are no implications between ones where no vectors are drawn, and no converses of the implications described above do not hold.

Outline of proof. We can see that (DEH) → (DVH) → (EH) by Fig. 2.1.

To show that (Iso) → (DVH), we use the next claim.

Claim. Each of the moves as illustrated in Fig. 2.2 is realized by delta moves and ambient isotopies.

Proof. See Fig. 2.3 for example.

□

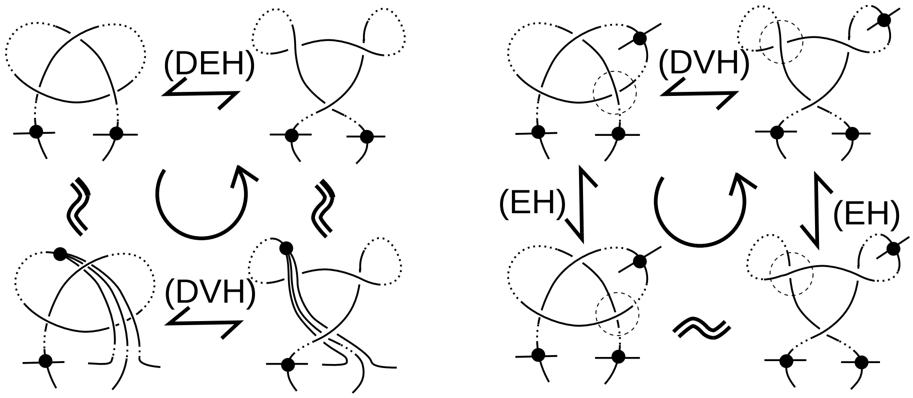


Fig. 2.1

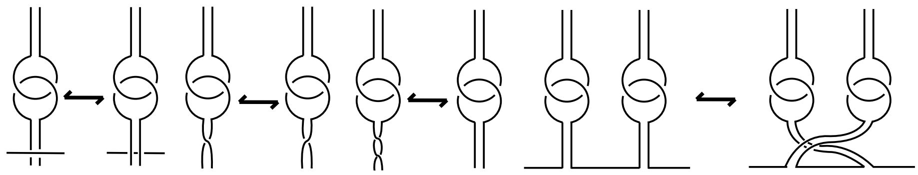


Fig. 2.2

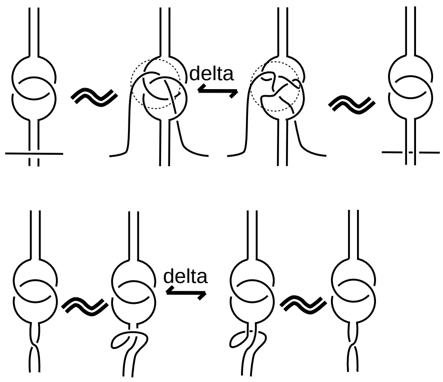


Fig. 2.3

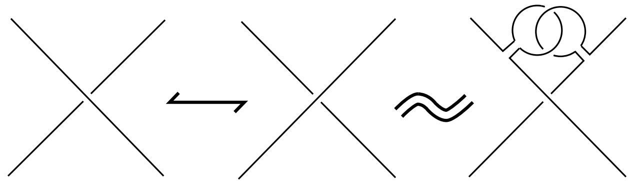


Fig. 2.4

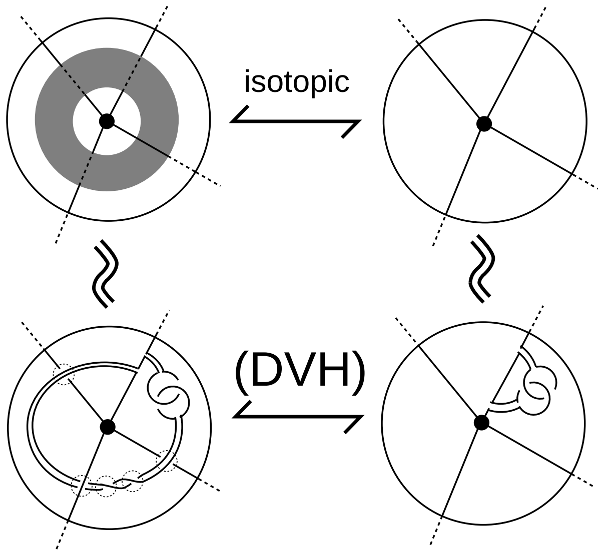


Fig. 2.5

We note that crossing changes can be regarded as a 'band sum of Hopf links', see Fig. 2.4. With the claim above and the fact that isotopy implies edge-homotopy, Fig. 2.5 illustrates the implication (Iso) → (DVH).

Please see [18] for detailed proofs and proofs of other parts. □

Together with ∽'s result [32], we have the following corollary.

Corollary 2.2. Let G be a finite graph. Then the following are equivalent.

(1) Every pair of spatial embeddings of G are isotopic.

(2) Every pair of spatial embeddings of G are I-equivalent.

(3) Every pair of spatial embeddings of G are delta vertex-homotpic.

(4) Every pair of spatial embeddings of G are edge-homotpic.

(5) G is a generalized bouquet, i.e., G does not contain a subgraph which is homeomorphic to disjoint cycles, K_4 or D_3 as illustrated in Fig. 2.6.

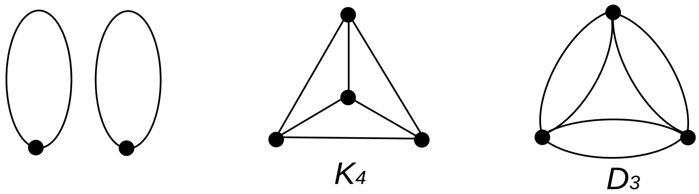


Fig. 2.6

Recently the author obtained the following related result.

Theorem 2.3. ([21]) Let G be a finite graph. Then the following are equivalent.

(1) Every pair of spatial embeddings of G are delta edge-homotpic.

(2) G does not contain a subgraph which is homeomorphic to a θ-curve or disjoint cycles.

(3) G is a bouquet as illustrated in Fig. 2.7.

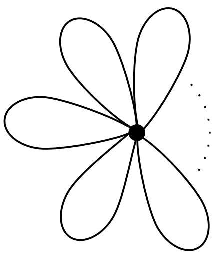


Fig. 2.7

3. Delta edge- & Delta vertex-homotopy invariants

We start with an example of a delta edge-homotopy invariant. A subgraph of G is called a *cycle* if it is homeomorphic to S^1. A cycle of a graph is called a *k-cycle* if it contains exactly k edges.

Example. Set G denotes a θ-curve, and give labels to its vertices and edges as in Fig. 3.1. We denote the three cycles $e_2 \cup e_3$, $e_3 \cup e_1$ and $e_1 \cup e_2$ of G by γ_1, γ_2 and γ_3 respectively.

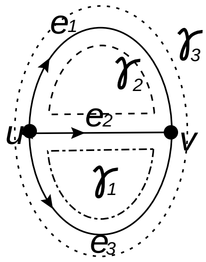


Fig. 3.1

Given spatial embedding $f : G \to S^3$, we define

$$\widetilde{\alpha}(f) \equiv \sum_{i=1}^{3} a_2(f(\gamma_i)) \pmod 2$$

where a_2 denotes the second coefficient of the *Conway polynomial* of the knot. This $\widetilde{\alpha}$ become a delta edge-homotopy invariant, showed as follows. Recall that $a_2(K_+) - a_2(K_-) = 1$ holds for the knots K_+ and K_- as illustrated in Fig. 3.2 [26].

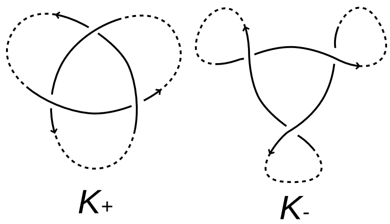


Fig. 3.2

Assume that a θ-curve g is obtained from f by a single self delta move on $f(e_1)$ as illustrated in Fig. 3.3. Then we have that

$$\widetilde{\alpha}(f) - \widetilde{\alpha}(g) \equiv \sum_{i=2}^{3} \{a_2(f(\gamma_i)) - a_2(g(\gamma_i))\} = 2 \equiv 0 \pmod 2.$$

This implies that $\widetilde{\alpha}$ is a delta edge-homotopy invariant.

Now let h be a trivial θ-curve and f a θ-curve as illustrated in Fig. 3.4. Then, by direct calculations, we have that $\widetilde{\alpha}(h) \equiv 0$ and $\widetilde{\alpha}(f) \equiv 1 \pmod 2$. Thus we can conclude that $h \overset{(\mathrm{DEH})}{\not\sim} f$. However it depends only upon the *Arf invariant* of constituent knots, so, it seems to be not strong enough. For example, we cannot distinguish any *almost trivial theta curve* (i.e., spatial embedding with no non-trivial knots) from the trivial one by $\widetilde{\alpha}$.

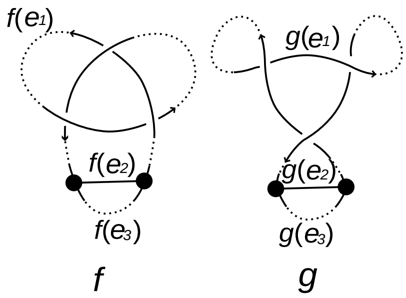


Fig. 3.3

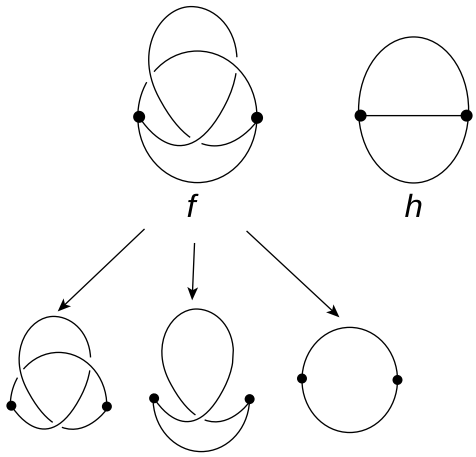


Fig. 3.4

We generalize this invariant in the following way. Let $\Gamma(G)$ be the set of all cycles of a graph G and $E(G)$ the set of all edges of G. For an edge $e \in E(G)$, we set

$$\Gamma_e(G) := \{\gamma \in \Gamma(G) \mid \gamma \supset e\}.$$

Also, for edges $e_1, e_2 \in E(G)$, we set

$$\Gamma_{e_1,e_2}(G) := \{\gamma \in \Gamma(G) \mid \gamma \supset e_1, e_2\}.$$

Definition 3.1. Let $\omega : \Gamma(G) \to \mathbf{Z}_m$ be a map, which we call a *weight*, where $\mathbf{Z}_m$ denotes the subset $\{0, 1, \ldots, m-1\}$ of the infinite cyclic group $\mathbb{Z}$ (we admit $m = 0$, and then $\mathbf{Z}_0$ denotes $\mathbb{Z}$).

(1) ω is *weakly balanced on an edge* $e \in E(G)$ if

$$\sum_{\gamma \in \Gamma_e(G)} \omega(\gamma) \equiv 0 \pmod{m}.$$

is satisfied.

(2) ω is *weakly balanced on a pair of adjacent edges* $e_1, e_2 \in E(G)$ if

$$\sum_{\gamma \in \Gamma_{e_1,e_2}(G)} \omega(\gamma) \equiv 0 \pmod{m}$$

is satisfied.

Given spatial embedding $f : G \to S^3$ and a weight $\Gamma(G) \to \mathbf{Z}_m$, we define

$$\widetilde{\alpha}_\omega(f) \equiv \sum_{\gamma \in \Gamma(G)} \omega(\gamma) a_2(f(\gamma)) \pmod{m}.$$

Theorem 3.2. Let $f : G \to S^3$ be a spatial embedding of a finite graph G and $\Gamma(G) \to \mathbf{Z}_m$ a weight.

(1) If ω is weakly balanced on every edge in $E(G)$, then $\widetilde{\alpha}_\omega$ is a delta edge-homotopy invariant.

(2) If ω is weakly balanced on every pair of adjacent edges in $E(G)$, then $\widetilde{\alpha}_\omega$ is a delta vertex-homotopy invariant.

The proof is similar to the discussion given in the first example in this section, and so we omit it. The invariant $\widetilde{\alpha}_\omega$ is an extension of $\widetilde{\alpha}$: In fact $\widetilde{\alpha}$ is obtained by choosing a weight $\omega : \Gamma(G) \to \mathbf{Z}_2$ with $\omega(\gamma) = 1$ for any $\gamma \in \Gamma(G)$ if G is a θ-curve.

Next we will construct another invariant by using an order three Vassiliev invariant of knots. In the following the graph G is assumed to be oriented, i.e., each edge of G are assumed to be oriented. We set the orientation of each cycle $\gamma \in \Gamma_e(G)$ induced from that of e. Also we set the orientation of each cycle $\gamma \in \Gamma_{e_1,e_2}(G)$ induced from that of e_1.

Definition 3.3 ([33])**.** Let $\omega : \Gamma(G) \to \mathbf{Z}_m$ be a weight.

(1) ω is *balanced on an edge* $e \in E(G)$ if

$$\sum_{\gamma \in \Gamma_e(G)} \omega(\gamma) \cdot \gamma = 0 \text{ in } H_1(G; \mathbb{Z}/m\mathbb{Z})$$

is satisfied.

(2) ω is *balanced on a pair of adjacent edges* $e_1, e_2 \in E(G)$ if

$$\sum_{\gamma \in \Gamma_{e_1,e_2}(G)} \omega(\gamma) \cdot \gamma = 0 \text{ in } H_1(G; \mathbb{Z}/m\mathbb{Z})$$

is satisfied.

Remark. If a weight ω is balanced on every edge in $E(G)$, the invariant $\widetilde{\alpha}_\omega$ is equal to the *α-invariant* α_ω in [33]. Also note that a balanced weight is weakly balanced.

Now, for a spatial embedding $f : G \to S^3$, we define

$$n_\omega(f) \equiv \frac{1}{18} \sum_{\gamma \in \Gamma(G)} \omega(\gamma) \cdot V^{(3)}_{f(\gamma)}(1) \pmod{m},$$

where $V_L(t)$ denotes the *Jones polynomial*[2] of the link L and

$$V^{(3)}_L(1) = \left.\frac{d^3}{dt^3}\right|_{t=1} V_L(t).$$

Remark. As we will say later, the value $\frac{1}{18}V^{(3)}_K(1)$ is always an integer. It is known that this $\frac{1}{18}V^{(3)}_K(1)$ is a basis of Vassiliev invariants of knots of order three.

Theorem 3.4. Let $f : G \to S^3$ be a spatial embedding of a finite graph G and $\Gamma(G) \to \mathbf{Z}_m$ a weight.

(1) If ω is balanced on every edge in $E(G)$, then n_ω is a delta edge-homotopy invariant.

(2) If ω is balanced on every pair of adjacent edges in $E(G)$, then n_ω is a delta vertex-homotopy invariant.

Key of the proof. Let K_+, K_- and K_0 be two knots and a 3-component link as illustrated in Fig. 3.5.

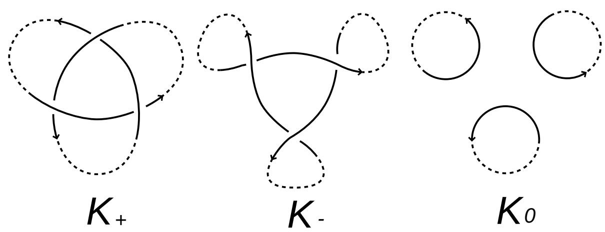


Fig. 3.5

Then we have that

$$\frac{1}{18}V^{(3)}_{K_+}(1) - \frac{1}{18}V^{(3)}_{K_-}(1) = 2\mathrm{Lk}(K_0) - 1,$$

where Lk denotes the *total linking number* (i.e., sum of pairwise linking numbers) of the link K_0. This is a corollary of more general formula obtained in [5]. Since the delta move is an unknotting operation, this formula implies that $\frac{1}{18}V^{(3)}_K(1)$ is an integer for any knot K.

Using this, the theorem follows from direct calculations. □

[2] We calculate the Jones polynomial of a link by the skein relation $tV_{J_+}(t) - t^{-1}V_{J_-}(t) = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})V_{J_0}(t)$.

Remark. The original idea of the theorem above is the construction of an edge-homotopy invariant of spatial graphs in [33] by using the formula $a_2(J_+)-a_2(J_-) = lk(K_0)$, where J_+, J_- and J_0 are two knots and a 2-component link as illustrated in Fig. 3.6.

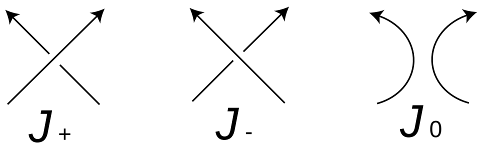


Fig. 3.6

Example 3.5. Let G be K_4 and $m \in \mathbb{Z}$. Let f_m be a spatial embedding of G as illustrated in Fig. 3.7 and h the trivial one.

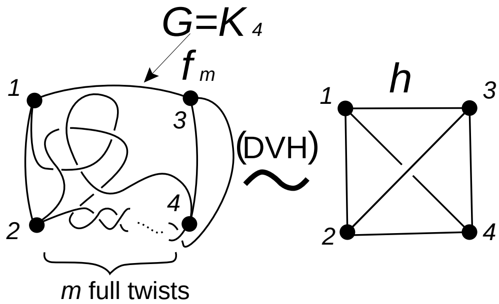


Fig. 3.7

Let $\omega_1 : \Gamma(K_4) \to \mathbf{Z}_4$ be the weight defined by $\omega(\gamma) = 1$ for every cycle $\gamma \in \Gamma(K_4)$. It is easily checked that this ω_1 is weakly balanced on every edge in $E(K_4)$.

The image $f_m(K_4)$ contains two non-trivial knots J_1 and J_2 as illustrated in Fig. 3.8.

We can calculate that $a_2(J_1) = 1$ and $a_2(J_2) = 1$. Thus we have

$$\widetilde{\alpha}_{\omega_1}(f_m) \equiv 2 \pmod 4$$

and also obviously we have

$$\widetilde{\alpha}_{\omega_1}(h) \equiv 0 \pmod 4.$$

These conclude that $f_m \overset{(\mathrm{DEH})}{\not\sim} h$ for any $m \in \mathbb{Z}$.

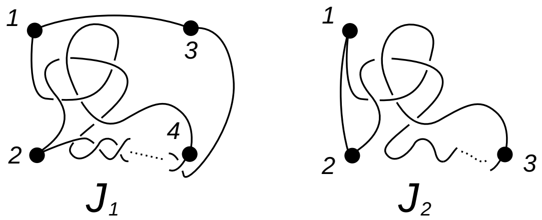


Fig. 3.8

Remark. It is easy to see that $f_m \overset{\text{(DVH)}}{\sim} h$ for any $m \in \mathbb{Z}$. Thus $\widetilde{\alpha}_{\omega_1}$ can detect the difference between delta edge-homotopy and delta vertex-homotopy.

Next we consider the weight $\omega_2 : \Gamma(K_4) \to \mathbb{Z}$ defined by

$$\Gamma(K_4) \ni \gamma \mapsto \begin{cases} 1 & \gamma: \text{ 3-cycle} \\ -1 & \gamma: \text{ 4-cycle.} \end{cases}$$

This ω_2 is also checked to be balanced on every edge in $E(K_4)$. By calculations, we obtain $V^{(3)}_{J_1}(1) = 36m - 18$ and $V^{(3)}_{J_2}(1) = -18$, and so,

$$n_{\omega_2}(f_m) = \frac{1}{18}(-18 - 36m + 18) = -2m.$$

This implies that $f_i \overset{\text{(DEH)}}{\not\sim} f_j$ for any $i \neq j$. So there are infinitely many spatial embeddings of K_4 up to delta edge-homotopy which are mutually delta vertex-homotopic.

Example 3.6. Let $G = K_5$ and $m \in \mathbb{N} \cup \{0\}$. Let f_m be a spatial embedding of G as illustrated in Fig. 3.9. Note that $f_m \overset{\text{(EH)}}{\sim} f_0$, which is achieved by the self crossing changes as in Fig. 3.10.

Let $\omega : \Gamma(K_5) \to \mathbb{Z}$ be the weight defined by

$$\Gamma(K_5) \ni \gamma \mapsto \begin{cases} 0 & \gamma: \text{ 3-cycle} \\ -1 & \gamma: \text{ 4-cycle} \\ 1 & \gamma: \text{ 5-cycle.} \end{cases}$$

Then we can check that this ω is balanced on every pair of adjacent edges in $E(K_5)$. After calculations, one can get $n_\omega(f_m) = -2m$, and therefore, $f_i \overset{\text{(DVH)}}{\not\sim} f_j$ for any $i \neq j$. By Theorem 2.1, this also implies that $f_i \overset{\text{(Isot)}}{\not\sim} f_j$ for any $i \neq j$.

Since all f_m's are mutually edge-homotopic, n_ω can detect the difference between edge-homotopy and delta vertex-homotpy. Moreover there are infinitely many spatial embeddings of K_5 up to delta vertex-homotopy (resp. isotopy) which are mutually edge-homotopic.

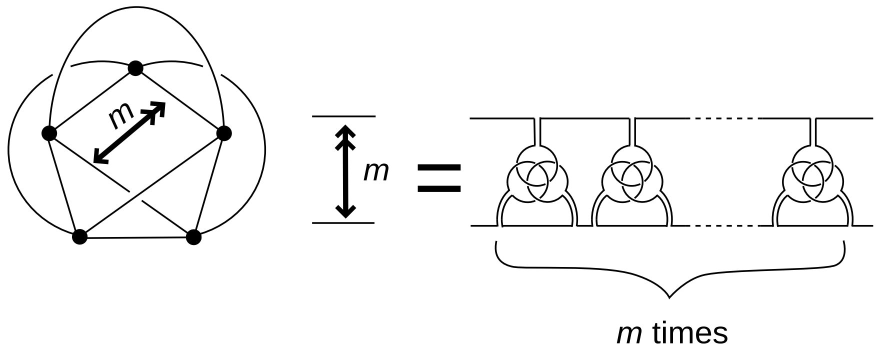


Fig. 3.9

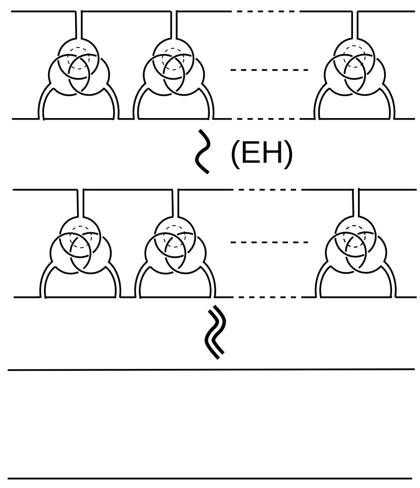


Fig. 3.10

4. Delta vertex- & Edge-homotopy classication of spatial embeddings of K_4

By Corollary 2.2, disjoint cycles, K_4, and D_3 are the 'smallest' graphs admitting non-trivial spatial embeddings up to delta vertex-homotopy. This also holds for isotopy, I-equivalence and edge-homotopy. Thus it is natural to ask:

Question. Can we classify spatial embeddings of such 'smallest' graphs up to delta vertex-homotopy?

We note that if G is the disjoint cycles then delta vertex-homotopy coinsides with edge-homotopy and all spatial embeddings of G can be classified completely by the linking number [17]. In this section we give an answer to the case of K_4.

Now we put $G = K_4$ or D_3 with labels depicted as in Fig. 4.1.

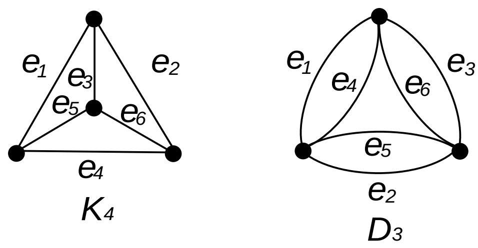


Fig. 4.1

To classify spatial embeddings of K_4 up to delta vertex-homotopy, we use the weight $\omega_2 : \Gamma(K_4) \to \mathbb{Z}$ defined in Example 3.5. To simplify the notation, here we use ω in stead of ω_2. This weight is checked to be balanced on every edge in $E(K_4)$.

Remark. The weight ω is actually shown to be the unique $\mathbb{Z}$-valued balanced weight up to multiplications of constant. Such a 'canonical' balanced weight exists for D_3, but here we omit the details.

Since ω is balanced, in particular, is weakly balanced, on every edge in $E(K_4)$, we can consider the invariant $\widetilde{\alpha}_\omega$. In this case, as noted in the remark just after Definition 3.3, the invariant $\widetilde{\alpha}_\omega$ is equal to $\backsim$'s α-invariant α_ω, which is an edge homotopy invariant.

To state our classification theorem, we need to introduce one more equivalence relation. An *adjacent-delta move* is defined to be the delta move on exactly three adjacent spatial edges as illustrated in Fig. 4.2.

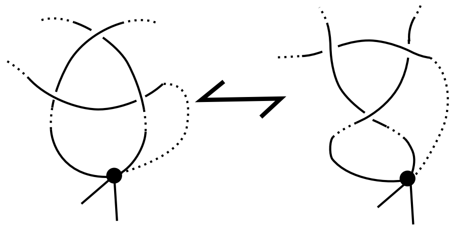


Fig. 4.2. Adjacent-delta move

Remark. An adjacent-delta move does not change the types of the knots included in the spatial graph.

Two spatial embedding $f, g : G \to S^3$ are Δ-*homotopic* if f and g are transformed into each other by quasi adjacent-delta moves, adjacent-delta moves and ambient isotopies. Then we have the following theorem.

Theorem 4.1. Let $G = K_4$ or D_3, and $\omega : \Gamma(G) \to \mathbb{Z}$ the 'canonical' balanced weight. Then two spatial embedding $f, g : G \to S^3$ are Δ-homotopic if and only if $\alpha_\omega(f) = \alpha_\omega(g)$.

It can be seen that, for a trivalent graph, an adjacent-delta move is realized by a sequence of quasi adjacent-delta moves (see Fig. 4.3). This indicates that, for a

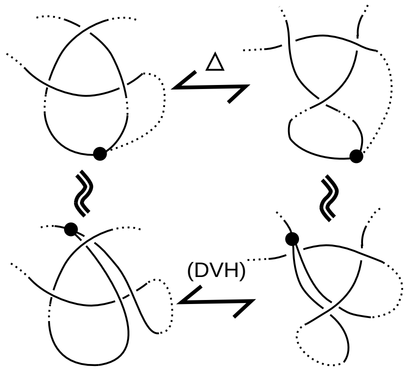


Fig. 4.3

trivalent graph G, in particular, for $G = K_4$, two spatial embedding $f, g : G \to S^3$ are Δ-homotopic only if $f \overset{(\mathrm{DVH})}{\sim} g$. Consequently we obtain:

Corollary 4.2. Let $f, g : K_4 \to S^3$ be two spatial embeddings of K_4. Then the following are equivalent.

(1) f and g are Δ-homotopic.
(2) $f \overset{\text{(DVH)}}{\sim} g$.
(3) $f \overset{\text{(EH)}}{\sim} g$.
(4) $\alpha_\omega(f) = \alpha_\omega(g)$.

Outline of the proof of Theorem 4.1. The 'only if' part is almost clear: It follows from the fact that α_ω depends essentially upon a_2 and that an adjacent-delta move does not change the types of the knots included in the spatial graph.

To prove the 'if' part, we construct a complete system of the representative for the Δ-homotopy types of spatial embeddings of G. For $G = K_4$, we prepare spatial embeddings h_m ($m \in \mathbb{Z}$) as illustrated in Fig. 4.4.

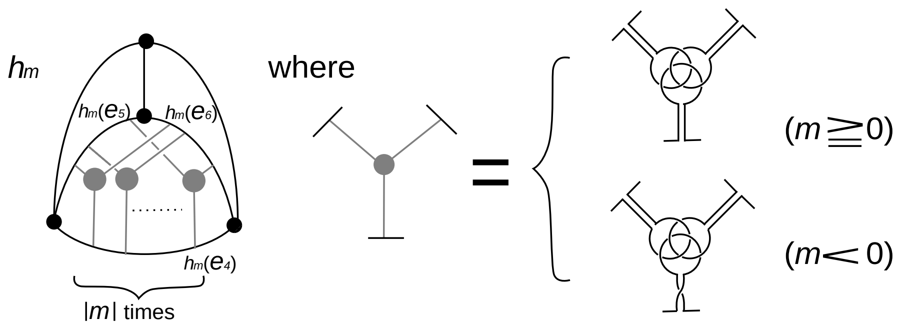


Fig. 4.4

Then we can show that if $\alpha_\omega(f) = m$, then f is Δ-homotopic to h_m. We refer the reader to [19] for the details. This completes the proof of the theorem. □

Problem. Classify spatial embeddings of D_3 up to delta vertex-homotopy or edge-homotopy.

Remark. We remark here that the α-invariant of a spatial embedding f of K_4 can be interpreted as Milnor's μ-invariant [10] of an associated 3-component link of f.

5. Delta edge-homotopy on θ-curves

By Theorem 2.3, disjoint cycles and a θ-curve are the 'smallest' graphs admitting non-trivial spatial embeddings up to delta edge-homotopy. Then again it is natural to ask the following.

Question. Can we classify spatial embeddings of disjoint cycles and a θ-curve up to delta edge-homotopy?

In the case of disjoint cycles, as we noted in Remark just after Definition 1.1, the question above is equivalent to the classification problem of 2-component links up to delta link-homotopy. This question was completely answered as follows.

Theorem 5.1 ([13, 14, 15])**.** Let $L = J_1 \cup J_2$ and $M = K_1 \cup K_2$ be oriented 2-component links. Then $L \overset{\text{(DEH)}}{\sim} M$ if and only if

(i) $\text{lk}(L) = \text{lk}(M)$ and

(ii) $a_3(L) - \text{lk}(L)\{a_2(J_1) + a_2(J_2)\} = a_3(M) - \text{lk}(M)\{a_2(K_1) + a_2(K_2)\}$.

As the result, it suffice to consider the case of a θ-curve. We prepare some terminology. We give labels to vertices, edges and cycles of a θ-curve, and give orientations to the edges as illustrated below. Given a spatial embedding f of a θ-curve, it is known [7] that there uniquely exists an orientable surface S_f such that S_f has the image of f as a spine and its Seifert linking form vanishes (i.e., all pairwise linking numbers of boundary curves are zero). Then we define the *associated 3-component link* L_f as the boundary $\partial S_f = K_f^1 \cup K_f^2 \cup K_f^3$, see Fig. 5.1.

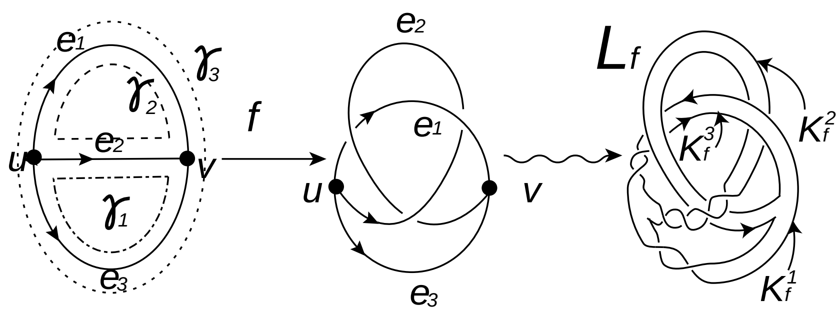


Fig. 5.1. Associated 3-component link

Note that each knot K_f^i is freely homotopic to $f(\gamma_i)$ for $i = 1, 2, 3$. Let $l_1(f) := K_f^2 \cup K_f^3$, $l_2(f) := K_f^3 \cup K_f^1$, $l_3(f) := K_f^1 \cup K_f^2$ be the 2-component sublinks of L_f. Note that these are all algebraically split links. Concerning these links the following is known.

Proposition 5.2 ([31],[4])**.** For every spatial embedding f of a θ-curve,

$$a_3(l_1(f)) = a_3(l_2(f)) = a_3(l_3(f))$$

holds.

Remark. Since the link $l_i(f)$ is algebraically split, we have that the *Sato-Levine invariant* [28] of $l_i(f)$ coincides with $a_3(l_i(f))$ [1].

By virtue of the proposition above, we can define $a_3(f)$ as $a_3(l_i(f))$ for some, and hence, any i. Now we can state our classification theorem.

Theorem 5.3. Let f and g be two spatial embeddings of a θ-curve. Then $f \overset{\text{(DEH)}}{\sim} g$ if and only if $a_3(f) = a_3(g)$.

Remark. By the result in [34], we can check that

$$a_3(f) \equiv \sum_{i=1}^{3} a_2(f(\gamma_i)) \equiv \widetilde{\alpha}(f) \pmod 2,$$

where the invariant $\widetilde{\alpha}(f)$ was defined in the first example in Section 3.

Example (Kinoshita's θ-curve)**.** Let f be the spatial embedding as illustrated in Fig. 5.2, called *Kinoshita's θ-curve.* This is an example of the almost unknotted theta curve.

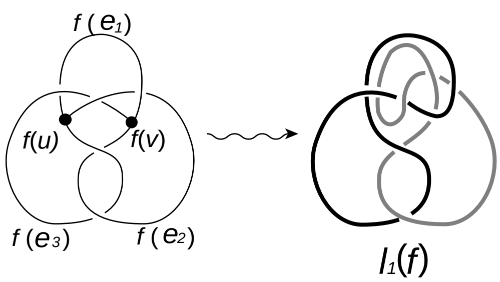


Fig. 5.2

For this f, $a_3(f) = a_3(l_1(f)) = 2$. Thus $f \overset{\text{(DEH)}}{\not\sim} h$, where h is the trivial embedding. Remark that the image of f does not contain nontrivial knots, and so, $\widetilde{\alpha}(f) = 0$.

Outline of the proof of Theorem 5.3. The 'only if' part follows from the fact that a_3 is invariant under self-delta moves on 2-component links.

To prove the 'if' part, we again construct a complete system of the representative for the delta edge-homotopy types of spatial embeddings of a θ-curves. We prepare spatial embeddings h_{m,ε_f}, where $m \in \mathbb{Z}$ and $\varepsilon_f \in \{0, 1\}$, as illustrated in Fig. 5.3. By a calculation we can see that $a_3(h_{m,\varepsilon_f}) = 2m + \varepsilon_f$.

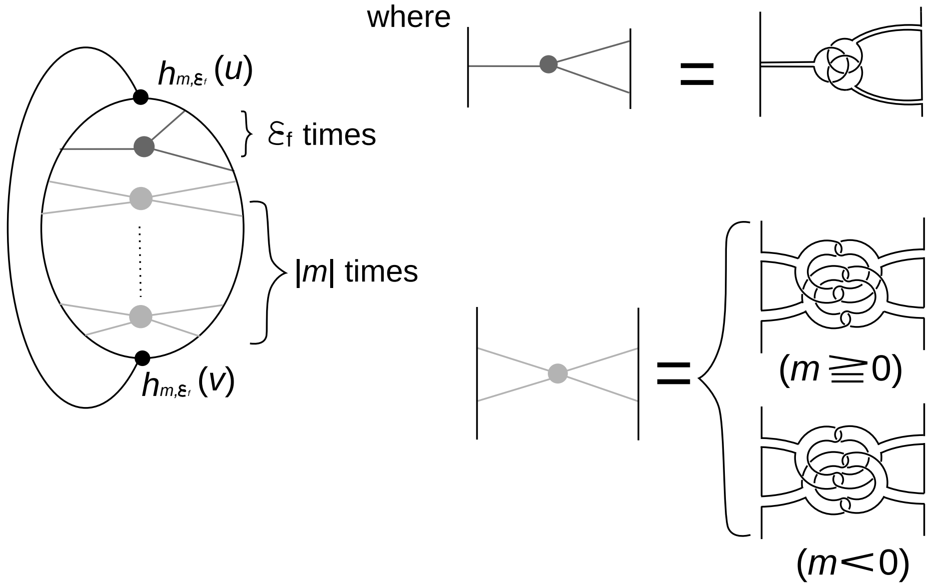


Fig. 5.3

Then we can show that there exists an integer m (resp. n) and $\varepsilon_f \in \{0,1\}$ (resp. $\varepsilon_g \in \{0,1\}$) such that f (resp. g) is delta edge-homotopic to h_{m,ε_f} (resp. h_{n,ε_g}). Thus by the assumption we have that

$$2m + \varepsilon_f = a_3(h_{m,\varepsilon_f}) = a_3(f) = a_3(g) = a_3(h_{n,\varepsilon_g}) = 2n + \varepsilon_g.$$

This implies that $\varepsilon_f = \varepsilon_g$ and $m = n$. Therefore we have that f and g are delta edge-homotopic. This completes the proof of the theorem. □

Problem. Classify spatial embeddings of K_4 up to delta edge-homotopy.

Finally we give some corollaries of Theorem 5.3 and discuss about related topics.

Corollary 5.4. Any boundary θ-curve is delta edge-homotopically trivial.

Here a spatial embedding f of a θ-curve θ is called a *boundary θ-curve* [25] if there exist compact, connected and orientable surfaces S_1, S_2 and S_3 in S^3 such that $S_i \cap f(\theta) = \partial S_i = f(\gamma_i)$ $(i = 1,2,3)$ and $\mathrm{int}S_i \cap \mathrm{int}S_j = \emptyset$ $(i \neq j)$.

Proof. For a boundary theta curve f, any 2-component sublink of L_f must be a boundary link. Since the Conway polynomial of any boundary link is zero [2], we have that $a_3(f) = 0$. Thus we have the result by Theorem 5.3. □

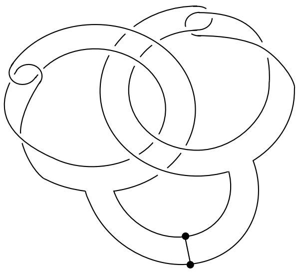


Fig. 5.4. Boundary θ-curve

As we noted in Remark just after Definition 1.1, it is known that any 2-component boundary link is delta edge-homotopically trivial. Thus the corollary above is the θ-curve version of this fact. We also note that the converse of the corollary above does not hold. Consider the spatial embedding as illustrated in Fig. 5.5. This is actually delta edge-homotopically trivial, but is not a boundary θ-curve [27].

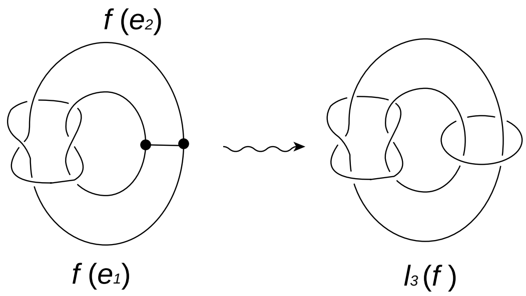


Fig. 5.5

Besides we mention the relationship between cobordism and delta edge-homotopy on spatial embeddings of a θ-curve. If two spatial embeddings f and g of a θ-curve are cobordant, then $l_i(f)$ and $l_i(g)$ are cobordant [31]. Thus we have the following by Theorem 5.3 and the cobordism invariance of the Sato-Levine invariant.

Corollary 5.5. Let f, g be two spatial embeddings of a θ-curve. Then $f \overset{\text{(Cob)}}{\sim} g$ implies $f \overset{\text{(DEH)}}{\sim} g$. In particular, any slice embedding (recall Definition 0.3(2)) of a θ-curve is delta edge-homotopically trivial.

It is known that such a implication does not exist for θ_n of $n \geq 4$.

The set of cobordism classes of spatial embeddings of a θ-curve admits a non-abelian group structure with the vertex connected sum [31, 11]. The corollary above suggests that the set of delta edge-homotopy classes, which we denote by DEH(θ), of spatial embeddings of a θ-curve might have a group structure.

This is actually true: DEH(θ) admits an abelian group structure with the vertex connected sum. Besides we have the following.

Theorem 5.6. The map $a_3 : \mathrm{DEH}(\theta) \stackrel{\cong}{\to} \mathbb{Z}$ yields an isomorphism. A generator of DEH(θ) is given by the theta curve f as in Figure 3.4.

Corollary 5.7. The set of delta edge-homotopy classes represented by almost trivial θ-curves is a subgroup of DEH(θ) isomorphic to $2\mathbb{Z}$ under a_3. A generator of DEH(θ) is given by Kinoshita's theta curve.

There are infinitely many spatial embeddings of a θ-curve up to cobordism which are almost trivial and delta edge-homotopically trivial. In fact the set of cobordism classes of almost trivial and delta edge-homotopically trivial spatial embeddings form a subgroup of the θ-curve cobordism group that contains $\mathbb{Z}^\infty$.

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