Resume for “Workshop of Fledglings on Low-Dimensional Topology”

Abstract. The braid index of a surface-knot $F$ is the minimum number among the degrees of all surface braids whose closures are ambient isotopic to $F$. We give a lower bound of the braid index of a surface-knot using the colorings by a quandle. As an application, we determine the braid indices of $S^2$-knots for infinitely many examples and give an infinite series of ribbon surface-knots of genus $g$ whose braid indices are $s + 2$ for each pair of integers $g \geq 0$ and $s \geq 1$.

1. Surface braids


Definition 1.1. (surface braid)
Let $D_2^1$ and $D_2^2$ be 2-disk and $X_m$ a fixed set of $m$ distinct interior points of $D_2^1$. Let $pr_i : D_2^1 \times D_2^2 \to D_2^1$ be the projection map to the $i$-th factor for each $i$ ($i = 1, 2$). A surface braid of degree $m$ (or surface $m$-braid) is a compact oriented surface $S$ embedded properly and locally flatly in $D_2^1 \times D_2^2$ such that
(i) the restriction map $pr_2|_S : S \to D_2^2$ is a branched covering map of degree $m$,
(ii) $\partial S = X_m \times \partial D_2^2$ ($\subset D_2^1 \times \partial D_2^2$), and
(iii) the branched covering $pr_2|_S$ is simple, that is, $|S \cap pr_2^{-1}(y)| = m - 1$ or $m$ for each $y \in D_2^2$.

Definition 1.2. (equivalence relation)
Two surface braids $S$ and $S'$ are said to be equivalent if there is an ambient isotopy $\{h_t\}_{t \in [0, 1]}$ such that
(i) $h_0 = \text{id}$, $h_1(S) = S'$,
(ii) for each $t \in [0, 1]$, $h_t$ is fiber-preserving, that is, there is a homeomorphism $h_t : D_2^1 \to D_2^1$ such that $pr_2 \circ h_t = h_t \circ pr_2$, and
(iii) for each $t \in [0, 1]$, $h_t|_{D_2^1 \times \partial D_2^2} = \text{id}$.

Definition 1.3. (closure)
Let $S^2$ be a 2-sphere obtained from $D_2^2$ attaching a 2-disk $\overline{D_2^2}$ along the boundary of $D_2^2$. A surface braid $S$ of degree $m$ is extended to a closed surface $\widehat{S}$ in $D_1^1 \times S^2$ ($\equiv D_1^1 \times (D_2^2 \cup \overline{D_2^2})$) such that
$\widehat{S} \cap (D_1^1 \times D_2^2) = S$ and $\widehat{S} \cap (D_1^1 \times \overline{D_2^2}) = X_m \times \overline{D_2^2}$.
Identifying $D^2 \times S^2$ with the tubular neighborhood of a standard 2-sphere in $\mathbb{R}^4$, we assume that $S$ is a closed oriented surface embedded in $\mathbb{R}^4$. We call it the closure of $S$ in $\mathbb{R}^4$.

**Theorem 1.4.** ([4, 10]) Any oriented surface-link in $\mathbb{R}^4$ is ambient isotopic to the closure of a surface braid of degree $m$ for some $m$.

**Definition 1.5.** The braid index of a surface-link $F$, denoted by $\text{Braid}(F)$, is the minimum number among the degrees of all surface braids whose closures are ambient isotopic to $F$.

**Remark 1.6.** (Known results about braid indices)

- The braid index of the trivial $n$-component $S^2$-link is $n$ ($n \geq 1$).
- $\text{Braid}(F) = 1 \iff F$: the trivial $S^2$-knot.
- $\text{Braid}(F) = 2 \iff F$: the trivial 2-component $S^2$-knot or the trivial $\Sigma_g$-knot ($g \geq 1$).
- $\text{Braid}(F) = 3 \iff F$: a ribbon surface-link. (it was shown that the other way does not hold.) ([5])
- There are infinitely many ribbon $S^2$-knots with braid index 3 ([5]).
- There are infinitely many ribbon $S^2$-knots with braid index 4 ([7]).

2. Quandles and Colorings

**Definition 2.1.** (Quandle)

A *quandle* $[1, 2]$ is a set $X$ equipped with a binary operation $(a, b) \mapsto a * b$ such that (i) $a * a = a$ for any $a \in X$, (ii) the map $*a : X \to X$ ($x \mapsto x * a$) is bijective for each $a \in X$, and (iii) $(a * b) * c = (a * c) * (b * c)$, for any $a, b, c \in X$. A function $f : X \to Y$ between quandles is a *homomorphism* if $f(a * b) = f(a) * f(b)$ for any $a, b \in X$. For each element $a \in X$, the map $*a : X \to X$ is a quandle automorphism of $X$ by (ii) and (iii), and we denote the inverse map $(*a)^{-1}$ by $\overline{a}$.

**Definition 2.2.** (Knot quandle and coloring)

For $n \geq 0$, let $M$ be an oriented $(n + 2)$-dimensional manifold and $L$ an oriented $n$-dimensional manifold embedded in $M$ properly and locally flatly. Let $N(L)$ denote a tubular neighborhood of $L$ in $M$. Take a fixed point $z \in E(L) = \text{Cl}(M \setminus N(L))$ and let $Q(M, L, z)$ be the set of homotopy classes of paths $\alpha : [0, 1] \to E(L)$ such that $\alpha(0) \in \partial E(L)$ and $\alpha(1) = z$. A point $p \in \partial E(L)$ lies on a unique meridional circle of $N(L)$. Let $m_p$ be the loop based at $p$ which goes along this meridional circle in a positive direction. The *knot quandle* of $L$ in $M$, with the base point $z$, is a quandle consisting of the set $Q(M, L, z)$ with a binary operation defined by

$$[\alpha] * [\beta] = [\alpha \cdot \beta^{-1} \cdot m_{\beta(0)} \cdot \beta].$$

When $M = \mathbb{R}^{n+2}$, we denote $Q(\mathbb{R}^{n+2}, L, z)$ by $Q(L)$ briefly.

Let $F$ be a surface-link and $X$ a finite quandle. A *coloring* of $F$ by $X$ is a quandle homomorphism $c : Q(F) \to X$ from the knot quandle $Q(F)$ to $X$. We denote by $\text{Col}_X(F)$ the set of all colorings of $F$ by $X$. Note that the number of the colorings, $|\text{Col}_X(F)|$, is an invariant of the surface-link $F$.

3. Main Results

**Theorem 3.1.** Let $F$ be a surface-link which is not a trivial $S^2$-link. Let $X$ be a finite quandle of order $N$, where $N$ is a positive integer. If the inequality $|\text{Col}_X(F)| > N^l$ holds for some positive integer $l$, then we have $\text{Braid}(F) \geq l + 2$. 
By using this theorem and Theorem 4.1, we determine the braid indices of $S^2$-knots for infinitely many examples.

**Theorem 3.2.** For an odd integer $n \geq 3$, let $K_n$ be the $S^2$-knot obtained from an $(n,2)$-torus knot by Artin's spinning construction. Then we have the following.

(i) The braid index of an $S^2$-knot $K_n(s)$ is $s + 2$, where $K_n(s)$ is the connected sum of $s$ copies of $K_n$.

(ii) The braid index of a $\Sigma_g$-knot $K_n(s,g)$ is also $s + 2$, where $K_n(s,g)$ is the connected sum of $K_n(s)$ and $g$ copies of a trivial $T^2$-knot.

**Theorem 3.3.** For each pair of integers $g \geq 0$ and $s \geq 1$, there exists an infinite series of ribbon surface-knots of genus $g$ whose braid indices are $s + 2$.

4. **Lemma, Proposition and Theorem**

**Theorem 4.1.** ([7]) If neither $F_1$ nor $F_2$ is a trivial $S^2$-knot, then the following inequality holds.

$$\text{Braid}(F_1 \# F_2) \leq \text{Braid}(F_1) + \text{Braid}(F_2) - 2$$

**Proposition 4.2.** Let $F$ be a surface-link which is not a trivial $S^2$-link. Let $\alpha$ be the minimum number of generators of the knot quandle $Q(F)$. Then we have $\text{Braid}(F) - 1 \geq \alpha$.

Let $R_m$ be a quandle consisting of the set $\{0, 1, \ldots, m-1\}$ with the binary operation defined by $i \ast j \equiv 2j - i \pmod{m}$, where $m$ is a positive integer. The quandle $R_m$ is called the dihedral quandle of order $m$.

**Lemma 4.3.** $|\text{Col}_{R_m}(K_n(s))| \leq m^{s+1}$.

The equality sign holds if and only if $n$ is divided by $m$.

**References**


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