

spatial graph とその invariant について

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Abstract

spatial graph, 特に trivalent vertex を持つ θ -curve や handcuff graph を分類する上では, その invariant が重要となります. invariant をいくつか紹介した後, 7 交点までの θ -curve の分類と 6 交点までの handcuff graph の分類を行ないます.

1. Yamada polynomial

In 1989, S. Yamada [19] introduced a polynomial invariant for spatial graph diagrams. Let g be a spatial graph diagram. Then the Yamada polynomial $R(g) \in \mathbf{Z}[x^{\pm 1}]$ is defined by the following recursive formulas:

- (1) $R(\emptyset) = 1$, where \emptyset denotes the empty graph,
- (2) $R\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right) = xR\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right) + x^{-1}R\left(\begin{array}{c} \diagdown \\ \diagup \end{array}\right) + R\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right)$,
- (3) $R\left(\begin{array}{c} \diagup \\ e \\ \diagdown \end{array}\right) = R\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right) + R\left(\begin{array}{c} \diagdown \\ \diagup \end{array}\right)$, where e is a nonloop edge,
- (4) $R(g_1 \sqcup g_2) = R(g_1)R(g_2)$, where $g_1 \sqcup g_2$ denotes the disjoint union of spatial graph diagrams g_1 and g_2 ,
- (5) $R(B_n) = -(-x - 1 - x^{-1})^n$, where B_n is the n -leafed bouquet. In particular, $R(\bullet) = R(B_0) = -1$, $R\left(\begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array}\right) = R(B_1) = x + 1 + x^{-1}$.

For a θ -curve diagram θ , we easily calculate $R(\theta)$ by using the following properties:

- (1) $R\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right) = -(x + 1 + x^{-1})(x + x^{-1}) = -x^2 - x - 2 - x^{-1} - x^{-2}$,
- (2) $R\left(\begin{array}{c} \bigcirc \\ \diagup \quad \diagdown \end{array}\right) = 0$,
- (3) $R(\theta' \sqcup \bigcirc) = (x + 1 + x^{-1})R(\theta')$ for an arbitrary θ -curve θ' ,
- (4) $R\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right) - R\left(\begin{array}{c} \diagdown \\ \diagup \end{array}\right) = (x - x^{-1})\left(R\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right) - R\left(\begin{array}{c} \diagdown \\ \diagup \end{array}\right)\right)$,
- (5) $R\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right) = x^2 R\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right)$, $R\left(\begin{array}{c} \diagdown \\ \diagup \end{array}\right) = x^{-2} R\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right)$,
- (6) $R\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right) = R\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right)$,

- $$(7) \quad R\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right) = R\left(\begin{array}{c} \diagdown \\ \diagup \end{array}\right),$$
- $$(8) \quad R\left(\begin{array}{c} \diagup \\ \diagup \end{array}\right) = R\left(\begin{array}{c} \diagdown \\ \diagdown \end{array}\right), \quad R\left(\begin{array}{c} \diagdown \\ \diagdown \end{array}\right) = R\left(\begin{array}{c} \diagup \\ \diagup \end{array}\right),$$
- $$(9) \quad R\left(-\begin{array}{c} \diagup \\ \diagdown \end{array}\right) = -xR\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right), \quad R\left(-\begin{array}{c} \diagdown \\ \diagup \end{array}\right) = -x^{-1}R\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right).$$

2. Prime basic 4-regular disk graphs

In 2001, H. Yamano [21] classified tangles of seven crossings or less in his master's thesis, where he used the concept of *prime basic 4-regular disk graphs*.

He obtained the following lemma.

Lemma 2.1. There exist six prime basic 4-regular disk graphs with up to seven vertices as in Fig. 1.

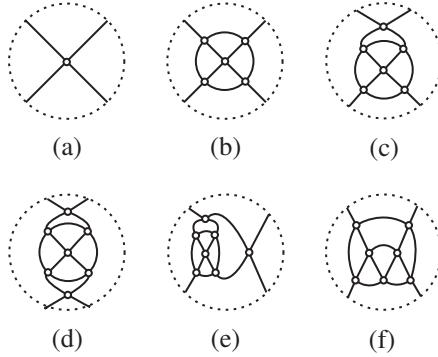


Fig. 1.

(a) P_1 (b) P_5 (c) P_6
 (d) P_{7-1} (e) P_{7-2} (f) P_{7-3}

3. Prime basic θ -polyhedra

From this section, we construct prime basic θ -polyhedra. A θ -polyhedron is a connected planar graph embedded in 2-sphere, whose two vertices are 3-valent, and the others are 4-valent. We can obtain a θ -curve diagram from a θ -polyhedron by substituting algebraic tangles for their 4-valent vertices.

A θ -polyhedron P_Θ is said to be *basic* if it contains no loop and no bigon.

Theorem 3.2. There exist seven type- \times prime basic θ -polyhedra with up to seven 4-valent vertices as in Fig. 2.

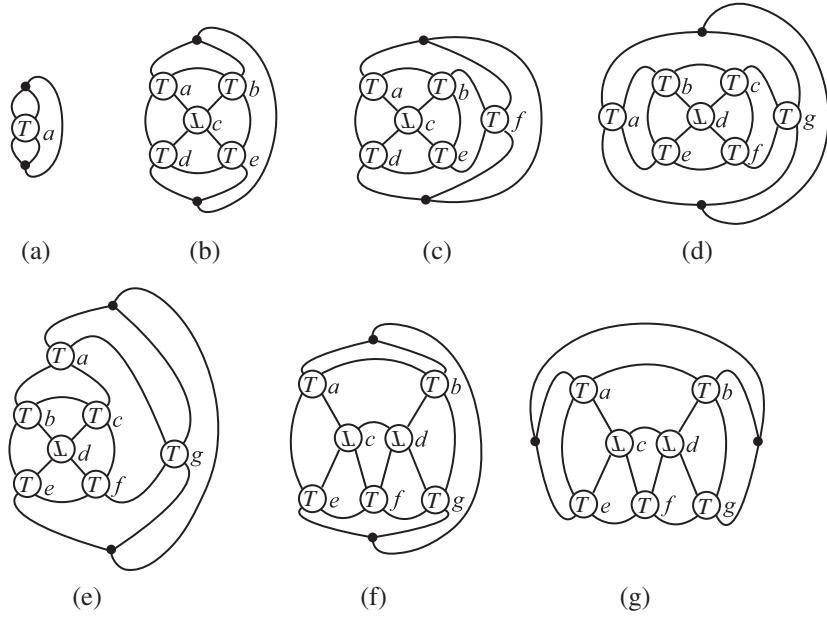


Fig. 2.

(a) $1_{\times}^1 a$	(b) $5_{\times}^1 a.b.c.d.e$	(c) $6_{\times}^1 a.b.c.d.e.f$	(d) $7_{\times}^1 a.b.c.d.e.f.g$
(e) $7_{\times}^2 a.b.c.d.e.f.g$	(f) $7_{\times}^3 a.b.c.d.e.f.g$	(g) $7_{\times}^4 a.b.c.d.e.f.g$	

Theorem 3.3. There exist seventeen type-* prime basic θ -polyhedra with up to seven 4-valent vertices as in Fig. 3.

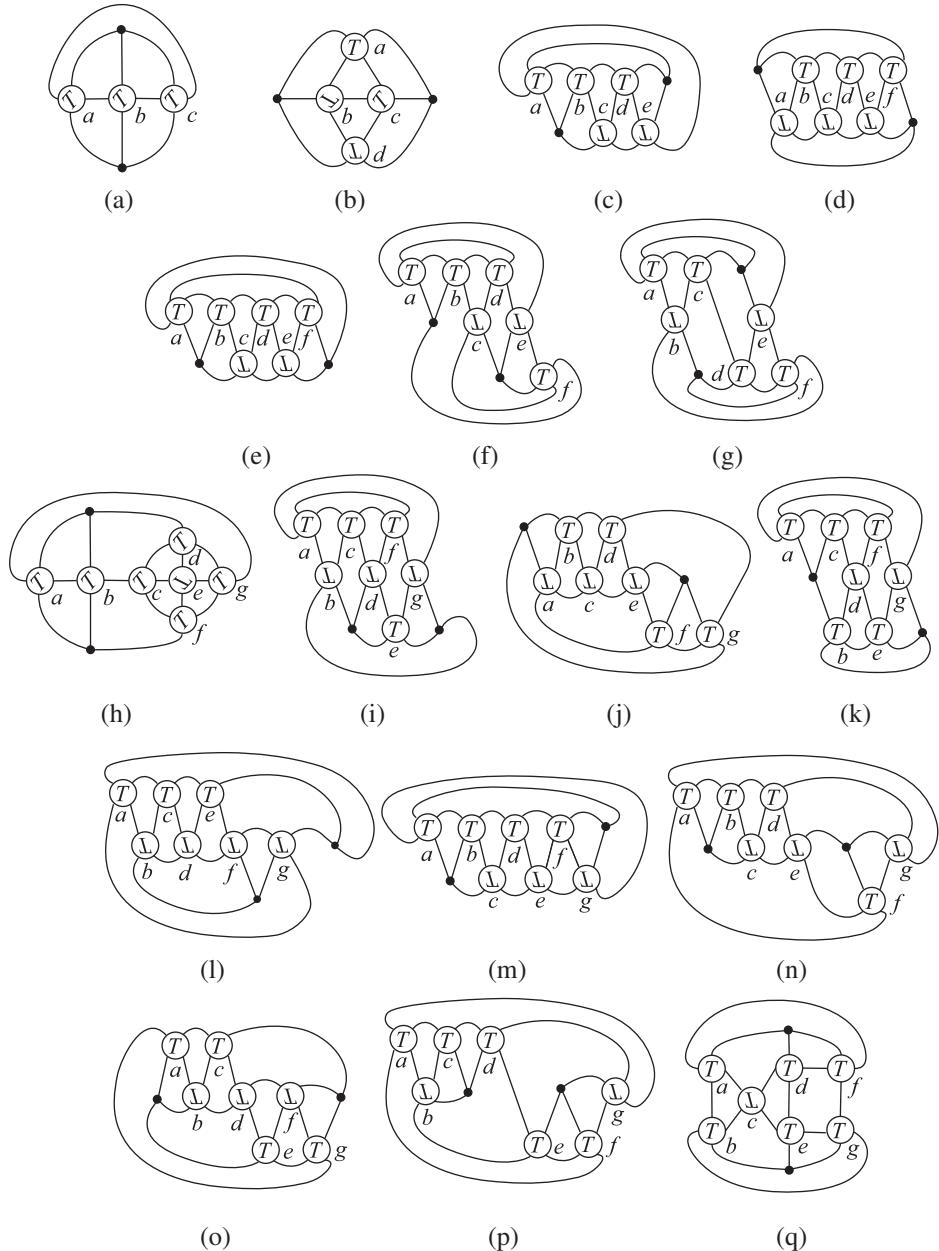


Fig. 3.

- | | | | |
|---------------------------|---------------------------|------------------------------|---------------------------|
| (a) 3^1_* a.b.c | (b) 4^1_* a.b.c.d | (c) 5^1_* a.b.c.d.e | (d) 6^1_* a.b.c.d.e.f |
| (e) 6^2_* a.b.c.d.e.f | (f) 6^3_* a.b.c.d.e.f | (g) 6^4_* a.b.c.d.e.f | |
| (h) 7^1_* a.b.c.d.e.f.g | (i) 7^2_* a.b.c.d.e.f.g | (j) 7^3_* a.b.c.d.e.f.g | (k) 7^4_* a.b.c.d.e.f.g |
| (l) 7^5_* a.b.c.d.e.f.g | (m) 7^6_* a.b.c.d.e.f.g | (n) 7^7_* a.b.c.d.e.f.g | |
| (o) 7^8_* a.b.c.d.e.f.g | (p) 7^9_* a.b.c.d.e.f.g | (q) 7^{10}_* a.b.c.d.e.f.g | |

4. Enumeration of θ -curve

We give an enumeration of θ -curve with up to seven crossings by using our notation. Knots in the second column correspond to Rolfsen's knot table [14], and θ -curves in the last column correspond to Litherland's table [11] (see Fig. 4–7). \overline{K} and $\overline{\Theta}$ denote mirror images of K and Θ , respectively.

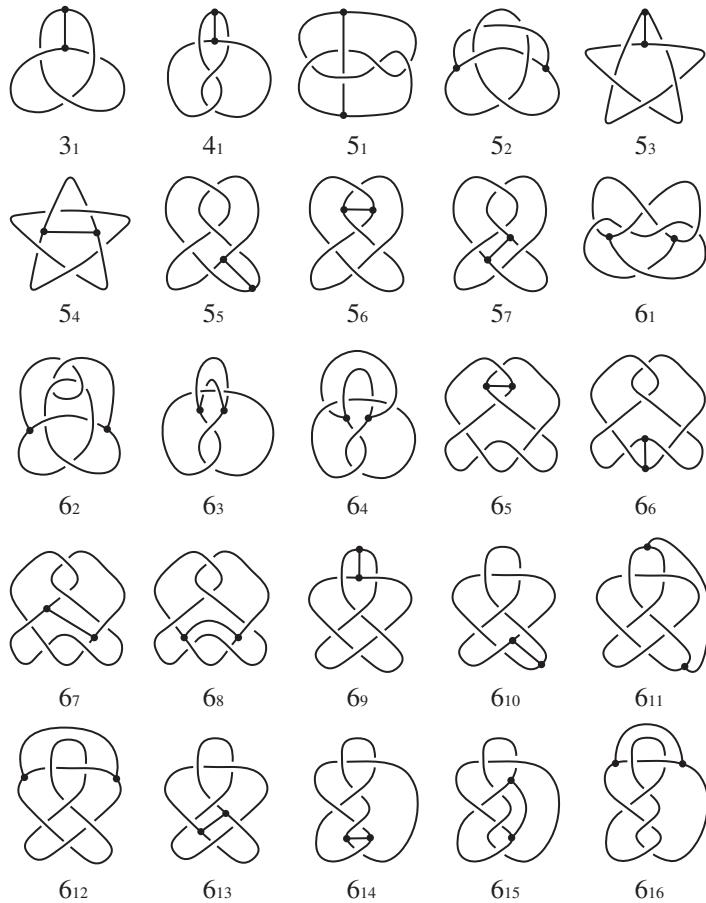


Fig. 4. Prime θ -curves with up to six crossings.

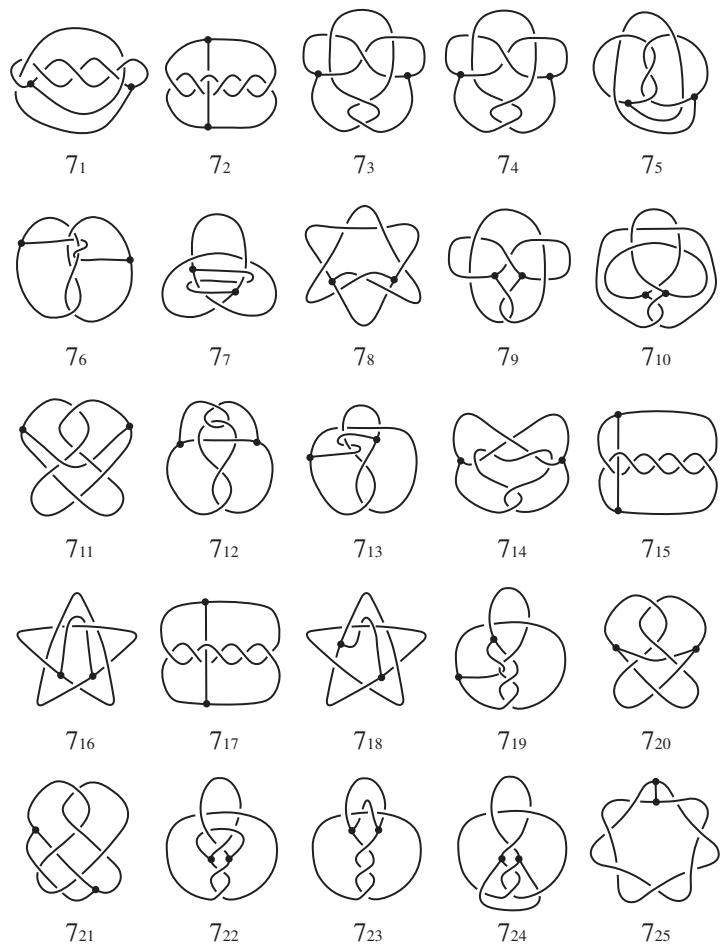


Fig. 5. Prime θ -curves with seven crossings.

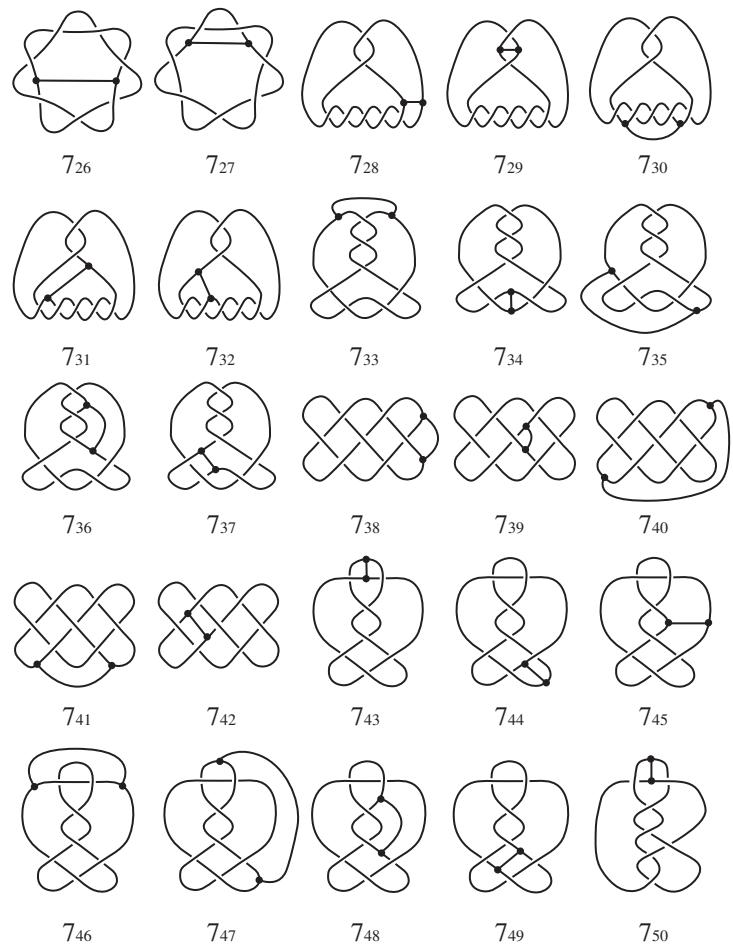


Fig. 6. Prime θ -curves with seven crossings (continued).

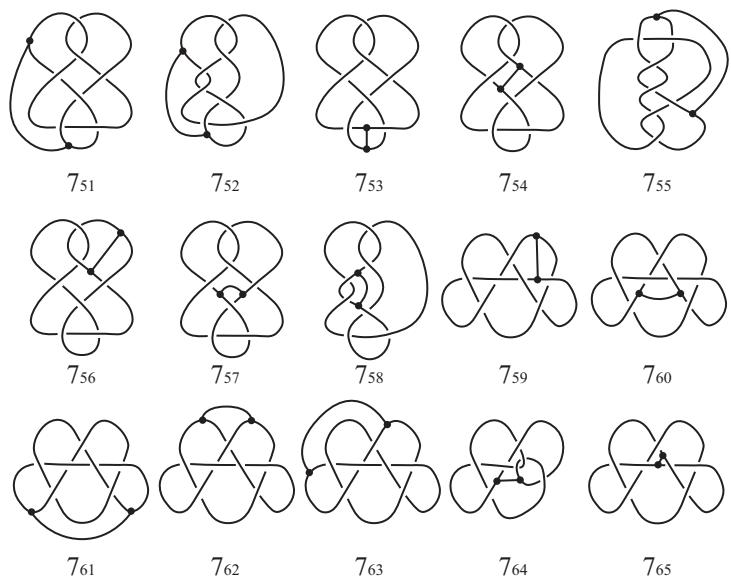


Fig. 7. Prime θ -curves with seven crossings (continued).

notation	constituent knot	θ -curve
$1_{\times}^1 3$	$3_1, 0, 0$	3_1
$1_{\times}^1 2 \ 2$	$4_1, 0, 0$	4_1
$1_{\times}^1 5$	$5_1, 0, 0$	5_3
$1_{\times}^1 3 \ 2$	$5_2, 0, 0$	5_6
$1_{\times}^1 2 \ 3$	$\overline{5}_2, 0, 0$	$\overline{5}_5$
$1_{\times}^1 3, 2$	$\overline{5}_1, \overline{3}_1, 0$	$\overline{5}_4$
$1_{\times}^1 2 \ 1, 2$	$5_2, 3_1, 0$	5_7
$3_*^1 2.2. - 1$	$\overline{3}_1, 0, 0$	5_2
$4_*^1 2.1.1.1$	$\overline{3}_1, 0, 0$	5_2
$4_*^1 2 \ 0.1.1.1$	$0, 0, 0$	5_1
$1_{\times}^1 4 \ 2$	$6_1, 0, 0$	6_5
$1_{\times}^1 2 \ 4$	$\overline{6}_1, 0, 0$	$\overline{6}_6$
$1_{\times}^1 3 \ 1 \ 2$	$6_2, 0, 0$	6_9
$1_{\times}^1 2 \ 1 \ 3$	$6_2, 0, 0$	6_{10}
$1_{\times}^1 2 \ 1 \ 1 \ 2$	$6_3, 0, 0$	$\overline{6}_{14}$
$1_{\times}^1 2 \ 2, 2$	$6_1, 4_1, 0$	6_8
$1_{\times}^1 2 \ 1 \ 1, 2$	$\overline{6}_2, 4_1, 0$	$\overline{6}_{13}$
$1_{\times}^1 3, 2 \ 1$	$\overline{6}_1, 0, 0$	$\overline{6}_7$
$1_{\times}^1 3, 2 +$	$\overline{6}_2, \overline{3}_1, 0$	$\overline{6}_{12}$
$1_{\times}^1 2 \ 1, 2 +$	$6_3, 3_1, 0$	$\overline{6}_{16}$
$3_*^1 3.2. - 1$	$4_1, 3_1, 0$	6_4
$3_*^1 2.2.2$	$0, 0, 0$	5_1
$3_*^1 2.2. - 2$	$0, 0, 0$	6_1
$4_*^1 2 \ 1.1.1.1$	$0, 0, 0$	6_1
$4_*^1 2 \ 1 \ 0.1.1.1$	$\overline{3}_1, 0, 0$	6_2
$4_*^1 - 2 - 1 \ 0.1.1.1$	$4_1, \overline{3}_1, 0$	$\overline{6}_4$
$4_*^1 2.2 \ 0.1.1$	$4_1, 3_1, 0$	6_4
$4_*^1 2.1.1.2 \ 0$	$4_1, \overline{3}_1, 0$	$\overline{6}_3$
$4_*^1 2.1.1. - 2 \ 0$	$4_1, 3_1, 0$	6_4
$4_*^1 2 \ 0.2 \ 0.1.1$	$\overline{3}_1, 0, 0$	6_2
$4_*^1 - 2 \ 0.2 \ 0.1.1$	$0, 0, 0$	$\overline{6}_1$
$5_{\times}^1 2.1.1.1.1$	$6_3, 0, 0$	6_{15}
$5_{\times}^1 2 \ 0.1.1.1.1$	$\overline{6}_2, 0, 0$	$\overline{6}_{11}$
$5_*^1 2 \ 0.1.1.1.1$	$0, 0, 0$	$\overline{6}_1$
$5_*^1 1.1.1.1.2 \ 0$	$\overline{5}_2, \overline{3}_1, 0$	$\overline{5}_7$
$6_*^2 1.1.1.1.1.1$	$0, 0, 0$	$\overline{6}_1$
$6_*^4 1.1.1.1.1.1$	$0, 0, 0$	$\overline{5}_1$

notation	constituent knot	θ -curve
$1_{\times}^1 7$	$7_1, 0, 0$	$\overline{7}_{25}$
$1_{\times}^1 5 \ 2$	$7_2, 0, 0$	$\overline{7}_{29}$
$1_{\times}^1 4 \ 3$	$7_3, 0, 0$	$\overline{7}_{33}$
$1_{\times}^1 3 \ 4$	$\overline{7}_3, 0, 0$	$\overline{7}_{34}$
$1_{\times}^1 2 \ 5$	$\overline{7}_2, 0, 0$	$\overline{7}_{28}$
$1_{\times}^1 3 \ 2 \ 2$	$7_5, 0, 0$	$\overline{7}_{43}$
$1_{\times}^1 3 \ 1 \ 3$	$7_4, 0, 0$	$\overline{7}_{38}$
$1_{\times}^1 2 \ 2 \ 3$	$7_5, 0, 0$	$\overline{7}_{44}$
$1_{\times}^1 2 \ 2 \ 1 \ 2$	$7_6, 0, 0$	$\overline{7}_{53}$
$1_{\times}^1 2 \ 1 \ 2 \ 2$	$\overline{7}_6, 0, 0$	$\overline{7}_{50}$
$1_{\times}^1 2 \ 1 \ 1 \ 1 \ 2$	$7_7, 0, 0$	$\overline{7}_{59}$
$1_{\times}^1 5, 2$	$\overline{7}_1, \overline{5}_1, 0$	$\overline{7}_{27}$
$1_{\times}^1 4 \ 1, 2$	$\overline{7}_3, \overline{5}_1, 0$	$\overline{7}_{36}$
$1_{\times}^1 3 \ 2, 2$	$7_3, \overline{5}_2, 0$	$\overline{7}_{37}$
$1_{\times}^1 2 \ 3, 2$	$\overline{7}_2, \overline{5}_2, 0$	$\overline{7}_{32}$
$1_{\times}^1 3 \ 1 \ 1, 2$	$\overline{7}_4, \overline{5}_2, 0$	$\overline{7}_{42}$
$1_{\times}^1 2 \ 2 \ 1, 2$	$\overline{7}_5, \overline{5}_2, 0$	$\overline{7}_{49}$
$1_{\times}^1 (3, 2), 2$	$7_5, \overline{5}_1, 0$	$\overline{7}_{48}$
$1_{\times}^1 (3, 2), -2$	$5_1, \overline{5}_2, 0$	7_{18}
$1_{\times}^1 (3, -2), 2$	$6_2, 0, 0$	6_{11}
$1_{\times}^1 (3, -2), -2$	$\overline{6}_3, 0, 0$	6_{15}
$1_{\times}^1 (2 \ 1, 2), 2$	$\overline{7}_6, \overline{5}_2, 0$	$\overline{7}_{58}$
$1_{\times}^1 (2 \ 1, 2), -2$	$\overline{5}_1, \overline{5}_2, 0$	$\overline{7}_{18}$
$1_{\times}^1 (2 \ 1, -2), 2$	$6_3, 0, 0$	6_{15}
$1_{\times}^1 (2 \ 1, -2), -2$	$\overline{6}_2, 0, 0$	$\overline{6}_{11}$
$1_{\times}^1 4, 3$	$\overline{7}_1, \overline{3}_1, 0$	$\overline{7}_{26}$
$1_{\times}^1 3 \ 1, 3$	$7_3, \overline{3}_1, 0$	$\overline{7}_{35}$
$1_{\times}^1 2 \ 2, 3$	$7_2, 0, 0$	$\overline{7}_{30}$
$1_{\times}^1 4, 2 \ 1$	$7_2, \overline{3}_1, 0$	$\overline{7}_{31}$
$1_{\times}^1 3 \ 1, 2 \ 1$	$7_5, \overline{3}_1, 0$	$\overline{7}_{45}$
$1_{\times}^1 2 \ 1 \ 1, 2 \ 1$	$\overline{7}_6, 0, 0$	7_{54}
$1_{\times}^1 2 \ 2, 2+$	$\overline{7}_6, 4_1, 0$	7_{57}
$1_{\times}^1 2 \ 2, -2-$	$6_2, 4_1, 0$	6_{13}
$1_{\times}^1 2 \ 1 \ 1, 2+$	$\overline{7}_7, 4_1, 0$	$\overline{7}_{65}$
$1_{\times}^1 2 \ 1 \ 1, -2-$	$\overline{6}_1, 4_1, 0$	$\overline{6}_8$
$1_{\times}^1 3, 3+$	$\overline{7}_4, 0, 0$	$\overline{7}_{39}$
$1_{\times}^1 3, -3-$	$\overline{6}_1, 0, 0$	$\overline{6}_7$

notation	constituent knot	θ -curve
$1_{\times}^1 2 \ 1, 2 \ 1+$	$7_7, 0, 0$	7_{62}
$1_{\times}^1 2 \ 1, 2 \ 1-$	$\overline{3}_1, 0, 0$	5_2
$1_{\times}^1 2 \ 1, -2 - 1+$	$6_1, 0, 0$	6_7
$1_{\times}^1 2 \ 1, -2 - 1-$	$\overline{6}_1, 0, 0$	$\overline{6}_7$
$1_{\times}^1 3, 2 + +$	$\overline{7}_5, \overline{3}_1, 0$	7_{46}
$1_{\times}^1 3, -2 - -$	$6_3, \overline{3}_1, 0$	6_{16}
$1_{\times}^1 2 \ 1, 2 + +$	$7_6, 3_1, 0$	$\overline{7}_{56}$
$1_{\times}^1 2 \ 1, -2 - -$	$6_2, 3_1, 0$	6_{12}
$3_*^1 (3, 2).1. - 1$	$\overline{6}_2, 0, 0$	$\overline{6}_{11}$
$3_*^1 (3, -2).1. - 1$	$5_1, \overline{5}_2, 0$	$\overline{7}_{18}$
$3_*^1 (2 \ 1, 2).1. - 1$	$6_3, 0, 0$	6_{15}
$3_*^1 (2 \ 1, -2).1. - 1$	$5_1, 5_2, 0$	7_{18}
$3_*^1 4.2. - 1$	$\overline{5}_2, 0, 0$	$\overline{7}_{20}$
$3_*^1 3 \ 1.2. - 1$	$\overline{5}_1, 0, 0$	7_{15}
$3_*^1 2 \ 1 \ 1.2. - 1$	$5_2, 3_1, 0$	$\overline{7}_{22}$
$3_*^1 2 \ 1 \ 1 \ 0. - 2 \ 1$	$\overline{6}_2, \overline{3}_1, 0$	$\overline{6}_{12}$
$3_*^1 3 \ 2 \ 1. - 1$	$5_2, 4_1, 0$	$\overline{7}_{24}$
$3_*^1 2 \ 1.2 \ 1. - 1$	$\overline{5}_1, 0, 0$	7_{17}
$3_*^1 3.2.2$	$\overline{3}_1, 0, 0$	6_2
$3_*^1 3.2. - 2$	$4_1, 0, 0$	$\overline{7}_{13}$
$3_*^1 3. - 2.2$	$4_1, 0, 0$	$\overline{7}_{13}$
$3_*^1 3. - 2. - 2$	$3_1, 0, 0$	$\overline{7}_6$
$3_*^1 3 \ 0. - 2.2$	$4_1, \overline{3}_1, 0$	6_4
$3_*^1 3 \ 0. - 2. - 2$	$3_1, \overline{3}_1, 0$	7_8
$3_*^1 2 \ 1. - 2 \ 0.2 \ 0$	$\overline{3}_1, 0, 0$	6_2
$3_*^1 2 \ 1. - 2 \ 0. - 2 \ 0$	$\overline{3}_1, 0, 0$	7_7
$4_*^1 4.1.1.1$	$3_1, \overline{3}_1, 0$	$\overline{7}_8$
$4_*^1 4 \ 0.1.1.1$	$\overline{3}_1, 0, 0$	7_5
$4_*^1 - 4 \ 0.1.1.1$	$5_1, 0, 0$	$\overline{7}_{15}$
$4_*^1 3 \ 1.1.1.1$	$\overline{3}_1, 0, 0$	7_6
$4_*^1 3 \ 1 \ 0.1.1.1$	$0, 0, 0$	$\overline{7}_1$
$4_*^1 - 3 - 1 \ 0.1.1.1$	$5_2, 0, 0$	7_{20}
$4_*^1 2 \ 2.1.1.1$	$\overline{3}_1, 0, 0$	7_7
$4_*^1 2 \ 2 \ 0.1.1.1$	$3_1, \overline{3}_1, 0$	7_{10}
$4_*^1 - 2 - 2 \ 0.1.1.1$	$\overline{5}_2, \overline{3}_1, 0$	$\overline{7}_{23}$
$4_*^1 2 \ 1.2 \ 0.1.1$	$4_1, 0, 0$	$\overline{7}_{13}$
$4_*^1 2 \ 1 \ 0.2 \ 0.1.1$	$4_1, 4_1, 0$	7_{14}
$4_*^1 - 2 - 1 \ 0.2 \ 0.1.1$	$4_1, 0, 0$	7_{13}

notation	constituent knot	θ -curve
$4_*^1 3.1.1.2$	$5_1, 0, 0$	$\overline{7}_{16}$
$4_*^1 3.0.1.1.2$	$5_2, 4_1, 0$	$\overline{7}_{23}$
$4_*^1 -3.0.1.1.2$	$\overline{5}_2, 4_1, 0$	$\overline{\overline{7}}_{24}$
$4_*^1 2.1.1.1.2$	$\overline{5}_2, 0, 0$	$\overline{7}_{21}$
$4_*^1 3.1.1.2.0$	$4_1, 0, 0$	$\overline{7}_{11}$
$4_*^1 3.1.1.-2.0$	$\overline{5}_2, 0, 0$	$\overline{7}_{20}$
$4_*^1 3.0.1.1.2.0$	$0, 0, 0$	$\overline{7}_2$
$4_*^1 3.0.1.1.-2.0$	$\overline{5}_1, 0, 0$	$\overline{7}_{15}$
$4_*^1 -3.0.1.1.2.0$	$5_1, 0, 0$	$\overline{7}_{17}$
$4_*^1 2.1.0.1.1.2.0$	$4_1, 0, 0$	$\overline{7}_{12}$
$4_*^1 2.1.0.1.1.-2.0$	$5_2, 3_1, 0$	$\overline{7}_{22}$
$4_*^1 -2.-1.0.1.1.2.0$	$\overline{5}_2, 4_1, 0$	$\overline{7}_{24}$
$4_*^1 2.3.1.1$	$\overline{5}_1, 0, 0$	$\overline{7}_{15}$
$4_*^1 2.3.0.1.1$	$\overline{5}_2, 0, 0$	$\overline{7}_{20}$
$4_*^1 2.2.1.1.1$	$5_2, 3_1, 0$	$\overline{7}_{22}$
$4_*^1 2.0.3.1.1$	$\overline{3}_1, 0, 0$	$\overline{7}_5$
$4_*^1 -2.0.3.1.1$	$3_1, \overline{3}_1, 0$	$\overline{7}_8$
$4_*^1 2.0.3.0.1.1$	$0, 0, 0$	$\overline{7}_1$
$4_*^1 -2.0.3.0.1.1$	$3_1, 0, 0$	$\overline{7}_6$
$4_*^1 2.0.2.1.1.1$	$3_1, \overline{3}_1, 0$	$\overline{7}_{10}$
$4_*^1 -2.0.2.1.1.1$	$3_1, 0, 0$	$\overline{7}_7$
$4_*^1 2.2.2.1$	$5_2, 4_1, 0$	$\overline{7}_{24}$
$4_*^1 2.2.2.0.1$	$\overline{5}_1, 0, 0$	$\overline{7}_{17}$
$4_*^1 -2.2.2.0.1$	$\overline{5}_1, 0, 0$	$\overline{7}_{17}$
$4_*^1 2.2.1.2$	$\overline{5}_1, \overline{5}_2, 0$	$\overline{7}_{18}$
$4_*^1 2.2.1.2.0$	$5_2, 0, 0$	$\overline{7}_{19}$
$4_*^1 2.2.1.-2.0$	$4_1, 0, 0$	$\overline{7}_{13}$
$4_*^1 2.0.2.0.2.1$	$4_1, 4_1, 0$	$\overline{7}_{14}$
$4_*^1 -2.0.2.0.2.1$	$\overline{3}_1, \overline{3}_1, 0$	$\overline{7}_9$

notation	constituent knot	θ -curve
$5_{\times}^1 2 \ 1.1.1.1.1$	$\overline{7}_7, 0, 0$	$\overline{7}_{63}$
$5_{\times}^1 - 2 - 1.1.1.1.1$	$\overline{6}_1, 4_1, 0$	$\overline{6}_8$
$5_{\times}^1 2 \ 1 \ 0.1.1.1.1$	$\overline{7}_6, 0, 0$	$\overline{7}_{51}$
$5_{\times}^1 2.1.1.2.1$	$\overline{7}_7, 0, 0$	$\overline{7}_{60}$
$5_{\times}^1 2.1.1. - 2.1$	$6_2, 0, 0$	6_{11}
$5_{\times}^1 - 2.1.1. - 2.1$	$5_1, 0, 0$	$\overline{7}_{16}$
$5_{\times}^1 2.1.1.1.2$	$\overline{7}_7, \overline{3}_1, 0$	$\overline{7}_{64}$
$5_{\times}^1 2.1.1.1. - 2$	$6_2, 4_1, 0$	6_{13}
$5_{\times}^1 2.2 \ 0.1.1.1$	$\overline{7}_6, 0, 0$	$\overline{7}_{52}$
$5_{\times}^1 2. - 2 \ 0.1.1.1$	$\overline{5}_1, 0, 0$	$\overline{7}_{16}$
$5_{\times}^1 - 2.2 \ 0.1.1.1$	$6_3, 0, 0$	$\overline{6}_{15}$
$5_{\times}^1 - 2. - 2 \ 0.1.1.1$	$\overline{6}_2, 0, 0$	$\overline{6}_{11}$
$5_{\times}^1 2.1.2 \ 0.1.1$	$\overline{7}_6, \overline{3}_1, 0$	$\overline{7}_{55}$
$5_{\times}^1 2 \ 0.1.2 \ 0.1.1$	$\overline{7}_5, \overline{3}_1, 0$	$\overline{7}_{47}$
$5_{\times}^1 2 \ 0.1.1.2 \ 0.1$	$\overline{7}_4, \overline{3}_1, 0$	$\overline{7}_{41}$
$5_{\times}^1 2 \ 0.1.1.1.2 \ 0$	$\overline{7}_4, \overline{3}_1, 0$	$\overline{7}_{40}$
$5_{*}^1 - 1. - 3.1.1.1$	$\overline{5}_1, \overline{5}_2, 0$	$\overline{7}_{18}$
$5_{*}^1 1.1.1.1. - 3$	$5_1, 5_2, 0$	7_{18}
$5_{*}^1 3 \ 0.1.1.1.1$	$4_1, 0, 0$	$\overline{7}_{13}$
$5_{*}^1 - 3 \ 0.1.1.1.1$	$4_1, 0, 0$	$\overline{7}_{13}$
$5_{*}^1 - 1.3 \ 0.1.1.1$	$6_1, 4_1, 0$	6_8
$5_{*}^1 1.1.1.1.3 \ 0$	$\overline{6}_1, 4_1, 0$	$\overline{6}_8$
$5_{*}^1 - 1.2 \ 1.1.1.1$	$6_3, 0, 0$	$\overline{6}_{15}$
$5_{*}^1 1.1.1.1. - 2 - 1$	$\overline{5}_1, \overline{5}_2, 0$	$\overline{7}_{18}$
$5_{*}^1 - 2. - 2.1.1.1$	$\overline{5}_1, 0, 0$	$\overline{7}_{16}$
$5_{*}^1 - 2. - 1. - 1. - 1.2$	$\overline{5}_1, 0, 0$	$\overline{7}_{16}$
$5_{*}^1 - 1. - 2.1.2.1$	$5_1, 0, 0$	7_{16}
$5_{*}^1 - 1. - 2.1.1.2$	$\overline{5}_1, \overline{5}_2, 0$	$\overline{7}_{18}$
$5_{*}^1 - 1. - 2. - 1. - 1.2$	$\overline{5}_1, \overline{5}_2, 0$	$\overline{7}_{18}$
$5_{*}^1 - 1. - 1. - 2. - 1.2$	$5_1, 0, 0$	7_{16}
$5_{*}^1 - 2.2 \ 0.1.1.1$	$4_1, \overline{3}_1, 0$	6_3
$5_{*}^1 2.1.1.1.2 \ 0$	$4_1, \overline{3}_1, 0$	$\overline{6}_3$
$5_{*}^1 - 1. - 2.1.2 \ 0.1$	$\overline{5}_2, 0, 0$	$\overline{7}_{21}$
$5_{*}^1 - 1. - 2.1. - 2 \ 0.1$	$4_1, \overline{3}_1, 0$	6_3
$5_{*}^1 1.2.1.1.2 \ 0$	$6_2, 4_1, 0$	6_{13}
$5_{*}^1 - 1. - 2.1.1.2 \ 0$	$\overline{5}_1, \overline{5}_2, 0$	$\overline{7}_{18}$
$5_{*}^1 1.1.1.2.2 \ 0$	$\overline{6}_1, 0, 0$	$\overline{6}_7$

notation	constituent knot	θ -curve
$5_*^1 2 \ 0.2.1.1.1$	$\overline{3}_1, \overline{3}_1, 0$	$\overline{7}_9$
$5_*^1 2 \ 0. - 2.1.1.1$	$4_1, \overline{3}_1, 0$	$\overline{6}_3$
$5_*^1 - 2 \ 0.2.1.1.1$	$0, 0, 0$	$\overline{7}_4$
$5_*^1 - 2 \ 0. - 2.1.1.1$	$5_2, 0, 0$	$\overline{7}_{21}$
$5_*^1 2 \ 0.1.2.1.1$	$4_1, 0, 0$	$\overline{7}_{13}$
$5_*^1 - 2 \ 0.1.2.1.1$	$\overline{3}_1, 0, 0$	$\overline{7}_6$
$5_*^1 2 \ 0.1.1.2.1$	$3_1, 0, 0$	$\overline{\overline{7}}_6$
$5_*^1 - 2 \ 0.1.1.2.1$	$4_1, 0, 0$	$\overline{\overline{7}}_{13}$
$5_*^1 2 \ 0.1.1.1.2$	$0, 0, 0$	$\overline{\overline{7}}_4$
$5_*^1 2 \ 0.1.1.1. - 2$	$\overline{5}_2, 0, 0$	$\overline{\overline{7}}_{21}$
$5_*^1 - 2 \ 0.1.1.1.2$	$3_1, 3_1, 0$	$\overline{\overline{7}}_9$
$5_*^1 - 2 \ 0.1.1.1. - 2$	$4_1, 3_1, 0$	6_3
$5_*^1 - 1.2 \ 0.2.1.1$	$6_1, 0, 0$	6_7
$5_*^1 - 1.2 \ 0.1.1.2$	$\overline{6}_2, 4_1, 0$	$\overline{6}_{13}$
$5_*^1 - 1. - 2 \ 0. - 1. - 1.2$	$5_1, 5_2, 0$	7_{18}
$5_*^1 - 1. - 1.2 \ 0. - 1.2$	$4_1, 3_1, 0$	6_3
$5_*^1 - 1. - 1. - 2 \ 0. - 1.2$	$\overline{5}_2, 0, 0$	$\overline{\overline{7}}_{21}$
$5_*^1 - 1.2 \ 0.1.2 \ 0.1$	$\overline{6}_2, 0, 0$	$\overline{6}_{11}$
$5_*^1 - 1.2 \ 0.1. - 2 \ 0.1$	$\overline{5}_1, 0, 0$	$\overline{7}_{16}$
$5_*^1 - 1. - 1.2 \ 0.2 \ 0.1$	$0, 0, 0$	6_1
$5_*^1 - 1. - 1.2 \ 0. - 2 \ 0.1$	$3_1, 0, 0$	$\overline{6}_2$
$5_*^1 - 1. - 1. - 2 \ 0.2 \ 0.1$	$\overline{5}_1, 0, 0$	$\overline{7}_{16}$
$5_*^1 - 1. - 1. - 2 \ 0. - 2 \ 0.1$	$0, 0, 0$	6_1
$5_*^1 1.1.2 \ 0.1.2 \ 0$	$6_2, 0, 0$	6_{11}
$5_*^1 1.1. - 2 \ 0.1.2 \ 0$	$5_1, 0, 0$	7_{16}
$6_*^1 1.2.1.1.1.1$	$0, 0, 0$	7_2
$6_*^1 1.2 \ 0.1.1.1.1$	$4_1, 0, 0$	7_{12}
$6_*^1 1. - 2 \ 0.1.1.1.1$	$4_1, 3_1, 0$	6_3
$6_*^2 1.1.1.2.1.1$	$3_1, 0, 0$	$\overline{7}_6$
$6_*^2 2 \ 0.1.1.1.1.1$	$0, 0, 0$	$\overline{7}_3$
$6_*^2 - 2 \ 0.1.1.1.1.1$	$0, 0, 0$	$\overline{7}_4$
$6_*^2 1.2 \ 0.1.1.1.1$	$4_1, 0, 0$	$\overline{7}_{13}$
$6_*^2 1.1.2 \ 0.1.1.1$	$\overline{3}_1, \overline{3}_1, 0$	$\overline{7}_9$
$6_*^2 1.1. - 2 \ 0.1.1.1$	$4_1, \overline{3}_1, 0$	$\overline{6}_3$

notation	constituent knot	θ -curve
$6_*^3 2.1. - 1.1.1.1$	$3_1, 3_1, 0$	$3_1 \#_3 \overline{3}_1$
$6_*^3 2.1.1.1. - 1. - 1$	$4_1, 3_1, 0$	6_4
$6_*^3 1.1.2.1.1.1$	$6_3, 0, 0$	6_{15}
$6_*^3 1.1. - 2.1.1.1$	$5_2, 0, 0$	7_{21}
$6_*^3 1.1.2.1. - 1. - 1$	$0, 0, 0$	$\overline{6}_1$
$6_*^3 1.1. - 2.1. - 1. - 1$	$5_1, 0, 0$	7_{16}
$6_*^3 1.1.1.1.2. - 1$	$6_3, 0, 0$	6_{15}
$6_*^3 1.1.1.1. - 2. - 1$	$5_2, 0, 0$	7_{21}
$6_*^3 1.1. - 1.1.2.1$	$0, 0, 0$	$\overline{6}_1$
$6_*^3 1.1. - 1.1. - 2.1$	$5_1, 0, 0$	7_{16}
$6_*^3 2 0.1.1.1. - 1. - 1$	$\overline{3}_1, 0, 0$	6_2
$6_*^3 2 0.1. - 1.1.1.1$	$3_1, \overline{3}_1, 0$	$3_1 \#_3 \overline{3}_1$
$6_*^3 2. - 1.1. - 1. - 1. - 1$	$\overline{3}_1, 0, 0$	3_1
$6_*^3 2. - 1. - 1. - 1.1.1$	$0, 0, 0$	6_1
$6_*^3 1.2 0.1.1. - 1. - 1$	$3_1, \overline{3}_1, 0$	$3_1 \#_3 \overline{3}_1$
$6_*^3 1.2 0. - 1.1.1.1$	$\overline{3}_1, 0, 0$	6_2
$6_*^3 1. - 2 0.1.1. - 1. - 1$	$3_1, 0, 0$	$\overline{3}_1$
$6_*^3 1. - 2 0. - 1.1.1.1$	$0, 0, 0$	$\overline{6}_1$
$6_*^3 1.1.2 0.1. - 1. - 1$	$4_1, \overline{3}_1, 0$	$\overline{6}_3$
$6_*^3 1.1. - 2 0.1.1.1$	$0, 0, 0$	6_1
$6_*^3 1.1.1.1. - 2 0. - 1$	$0, 0, 0$	6_1
$6_*^3 1.1. - 1.1. - 2 0.1$	$4_1, \overline{3}_1, 0$	$\overline{6}_3$
$6_*^4 2 0.1.1.1.1.1$	$3_1, 0, 0$	$\overline{6}_2$
$6_*^4 1.2 0.1.1.1.1$	$4_1, \overline{3}_1, 0$	$\overline{6}_3$
$6_*^4 1.1.2 0.1.1.1$	$0, 0, 0$	$\overline{6}_1$
$7_*^4 1.1.1.1.1.1.1$	$7_7, 0, 0$	7_{61}
$7_*^3 1.1.1.1.1.1.1$	$6_3, 0, 0$	$\overline{6}_{15}$
$7_*^4 1.1.1.1.1.1.1$	$0, 0, 0$	7_4
$7_*^4 1. - 1.1. - 1. - 1.1. - 1$	$4_1, \overline{3}_1, 0$	$\overline{6}_3$
$7_*^9 1.1.1.1.1.1.1$	$\overline{3}_1, 0, 0$	6_2

5. Enumeration of handcuff graph

We give an enumeration of handcuff graph with up to six crossings by using our notation. Links in the second column correspond to Rolfsen's knot table [14], and handcuff graphs in the last column correspond to Fig. 8. \overline{L} and $\overline{\Phi}$ denote mirror images of L and Φ , respectively.

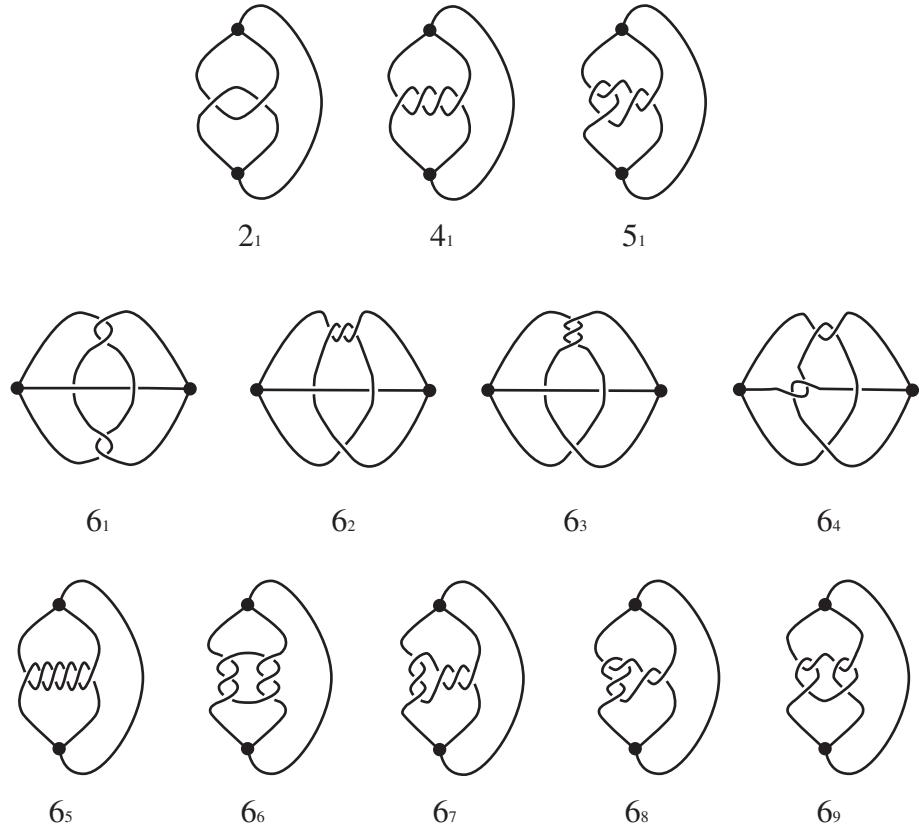


Fig. 8. Prime handcuff graphs with up to six crossings.

notation	constituent link	handcuff graph
$1_{\times}^1 2$	2_1^2	2_1
$1_{\times}^1 4$	4_1^2	4_1
$1_{\times}^1 2 1 2$	5_1^2	5_1
$1_{\times}^1 6$	6_1^2	6_5
$1_{\times}^1 3 3$	6_2^2	6_7
$1_{\times}^1 2 2 2$	6_3^2	6_8
$1_{\times}^1 3,3$	6_1^2	6_6
$1_{\times}^1 2 1,2 1$	6_3^2	6_9
$3_*^1 2 0. - 2.1$	4_1^2	4_1
$3_*^1 2 0.2. - 1$	0_1^2	$2_1 \#_3 \overline{2}_1$
$3_*^1 - 2 0. - 2 0.2$	2_1^2	6_2
$3_*^1 2 1 0.2 0. - 1$	4_1^2	$\overline{6}_4$
$4_*^1 3.1.1.1$	2_1^2	6_2
$4_*^1 3 0.1.1.1$	2_1^2	6_3
$4_*^1 - 3 0.1.1.1$	4_1^2	$\overline{6}_4$
$4_*^1 2.2.1.1$	4_1^2	6_4
$4_*^1 2 0.2.1.1$	2_1^2	6_3
$4_*^1 - 2 0.2.1.1$	2_1^2	$\overline{6}_2$
$4_*^1 2 0.1.1.2 0$	0_1^2	6_1
$4_*^1 2 0.1.1. - 2 0$	4_1^2	6_4
$6_*^1 1.1.1.1.1.1$	0_1^2	6_1
$6_*^3 1.1.1.1.1.1$	5_1^2	5_1^2
$6_*^3 1.1.1.1. - 1. - 1$	2_1^2	$2_1 \#_3 3_1 \theta$
$6_*^3 1.1. - 1.1.1.1$	2_1^2	$2_1 \#_3 3_1 \theta$

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