A note on generic fundamental polyhedra

牛島 顕

平成十六年 十月 十九日

1 Preliminaries

Let $\mathbb{H}^3 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|x_3 > 0\}$ be the *upper half-space model* of hyperbolic space. We regard its boundary in $\mathbb{R}^3 \cup \{\infty\}$ as $\mathbb{C} \cup \{\infty\}$, which is called the *sphere at infinity*, and we denote it by $\partial\mathbb{H}^3$. Let $\mathbb{B}^3$ be the open unit sphere centred at the origin of $\mathbb{R}^3$, and we regard it as the *Poincaré ball model* of hyperbolic space; indeed, an isometry, say Φ, from $\mathbb{H}^3$ to $\mathbb{B}^3$ is given by the reflection along $\mathbb{C}$, followed by the inversion along the sphere of radius $\sqrt{2}$ and centre $(0, 0, 1)$. Then $\Phi(\partial\mathbb{H}^3) = \partial\mathbb{B}^3$ and $\Phi|_{\mathbb{C}}$ coincide with the stereographic projection from $(0, 0, 1)$. For a hyperbolic plane P in $\mathbb{H}^3$, we denote its closure in $\overline{\mathbb{H}^3} := \mathbb{H}^3 \cup \partial\mathbb{H}^3$ by $\overline{P}$, and its boundary in $\overline{\mathbb{H}^3}$ (i.e., $\overline{P} \cap \partial\mathbb{H}^3$) by ∂P. Any hyperbolic plane in $\mathbb{B}^3$ is the image of some hyperbolic plane P in $\mathbb{H}^3$ by Φ, and it is the intersection with $\mathbb{B}^3$ of a Euclidean sphere or plane, say P^*, orthogonal to $\partial\mathbb{B}^3$. If P^* is a sphere, then we say its centre (in the Euclidean sense) the *centre* of P (or $\Phi(P)$), and if P^* is a plane, then the *centre* of P (or $\Phi(P)$) means ∞. Every point in $\mathbb{R}^3 - \overline{\mathbb{B}^3}$ uniquely determines a hyperbolic plane in $\mathbb{B}^3$.

Definition. 1. For a point $\boldsymbol{x} \in \overline{\mathbb{H}^3}$ and a loxodromic or a parabolic element $T \in \mathrm{PSL}_2(\mathbb{C})$, we define $B(\boldsymbol{x}; T) \subset \mathbb{H}^3$ as follows:

   (a) If $\boldsymbol{x} \in \mathbb{H}^3$ with $T(\boldsymbol{x}) \neq \boldsymbol{x}$, then $B(\boldsymbol{x}; T)$ is defined as the bisecting perpendicular geodesic plane of the geodesic segment joining $\boldsymbol{x}$ and $T^{-1}(\boldsymbol{x})$.

   (b) If $\boldsymbol{x} \in \partial\mathbb{H}^3$ with $T(\boldsymbol{x}) \neq \boldsymbol{x}$, then $B(\boldsymbol{x}; T)$ is a geodesic plane defined (and well-defined) as a limit of $B(\boldsymbol{x}'; T)$ by a convergence of $\boldsymbol{x}' \in \mathbb{H}^3$ to $\boldsymbol{x}$.

   (c) If $\boldsymbol{x}$ is a fixed point of T, then $B(\boldsymbol{x}; T) := \{\boldsymbol{x}\}$.

2. For a point $\boldsymbol{x} \in \overline{\mathbb{H}^3}$, we denote the centre of $\Phi(B(\boldsymbol{x}; T))$ by $C(\boldsymbol{x}; T)$.

The following lemma says that the definition of $B(\boldsymbol{x}; T)$ for a fixed point $\boldsymbol{x}$ is well-defined.

Lemma 1 ([JM88, Lemma 2.4]). *Let $T \in \mathrm{PSL}_2(\mathbb{C})$, and let $\boldsymbol{x} \in \partial\mathbb{H}^3$ with $T(\boldsymbol{x}) \neq \boldsymbol{x}$.*

1. *If T is loxodromic, then $\partial B(\boldsymbol{x};T)$ does not pass through any fixed point of T. As $\boldsymbol{x}$ approaches a fixed point of T, then $B(\boldsymbol{x};T)$ converges to the point.*

2. *If T is parabolic, then $\partial B(\boldsymbol{x};T)$ passes through the fixed point of T. There is a suitable approach of $\boldsymbol{x}$ to the fixed point such that $B(\boldsymbol{x};T)$ converges to the point.* □

When $\boldsymbol{x} \in \mathbb{H}^3$, the actual coordinate of $C(\boldsymbol{x};T)$ is obtained by the following calculations:

$$\begin{aligned}
\boldsymbol{a} &:= -\Phi(\boldsymbol{x}) + \left(1 - |\Phi(\boldsymbol{x})|^2\right)\left(\Phi(T^{-1}\boldsymbol{x})^* - \Phi(\boldsymbol{x})\right)^*, \\
\boldsymbol{b} &:= \boldsymbol{a}^* + \left(1 - |\boldsymbol{a}^*|^2\right)\left(\Phi(\boldsymbol{x}) + \boldsymbol{a}^*\right)^*, \\
C(\boldsymbol{x};T) &= \left\{\Phi(\boldsymbol{x}) + \left(1 - |\Phi(\boldsymbol{x})|^2\right)\left(\Phi(\boldsymbol{x}) + \boldsymbol{b}^*\right)^*\right\}^*,
\end{aligned}$$

where $\boldsymbol{y}^*$ means the complex conjugate of $\boldsymbol{y}^{-1}$. You can see that $C(\boldsymbol{x};T)$ converges to the expression in [JM88, 2.9] as $x_3 \to 0$, if you calculate $C(\boldsymbol{x};T)$ explicitly.

2 Notations and terminologies

1. First we define a Dirichlet fundamental polyhedron $\mathcal{P}_0(y)$ as follows:

$$\mathcal{P}_0(y) := \begin{cases} \left\{ x \in \mathbb{H}^3 \,\middle|\, d(x,y) \le d(y,Tx), \forall T \in G \right\} & \text{if } y \in \mathbb{H}^3, \\ \left\{ x \in \mathbb{H}^3 \,\middle|\, \begin{array}{l} x \text{ lies in the closure of the} \\ \text{component of } \mathbb{H}^3 - B(y;T) \text{ that} \\ \text{is adjacent to } y,\ \forall T \in G, T \ne \text{id.} \end{array} \right\} & \text{if } y \in \partial\mathbb{H}^3. \end{cases}$$

   - When $y \in \partial\mathbb{H}^3$, $\mathcal{P}_0(y)$ is well defined and is a fundamental polyhedron only when y has special properties, for example, when $y \in \Omega(G)$, where $\Omega(G)$ is the set of ordinary points with respect to G on $\partial\mathbb{H}^3$
   - The point y is called the *center* of $\mathcal{P}_0(y)$.
   - $\mathcal{P}(y) := \overline{\mathcal{P}_0(y)} \cap \left(\mathbb{H}^3 \cup \Omega(G)\right)$.

2. Associate with each edge e of $\mathcal{P}_0(y)$, is an *edge cycle of length k*, $(T_1 = \text{id.}, T_2, \ldots, T_k, T_{k+1} = T_1 = \text{id.})$, where $(\mathcal{P}(y), T_2\mathcal{P}(y), \ldots, T_k\mathcal{P}(y))$ is the cyclic arrangement of polyhedra about e in the G-orbit of $\mathcal{P}(y)$.

   Equivalently, k is the number of disjoint edges of $\mathcal{P}_0(y)$ that are equivalent to e under G.

3. - The *order* k of a vertex v of $\mathcal{P}_0(y)$ is the number of distinct vertices of $\mathcal{P}_0(y)$ that are equivalent to v under G.

Equivalently, k is the number of polyhedra,
$\mathcal{P}_0(y), T_2\mathcal{P}_0(y), \ldots, T_k\mathcal{P}_0(y)$
in the G-orbit of $\mathcal{P}(y)$ that share the vertex v.

- The transformations $T_i \in G$, are said to *be associated with* v.

4.
- A *cusp* of $\mathcal{P}(y)$ is a parabolic fixed point that lies in the Euclidean closure of $\mathcal{P}(y)$.
- It is of *rank one* or *rank two* according to the rank of the parabolic subgroup that fixes it.

5.
- *Boundary vertices* and *boundary edges* of $\mathcal{P}(y)$ are those lie in $\Omega(G)$.
- Associated with each boundary vertex is a *vertex cycle* analogous to the edge cycles of $\mathcal{P}_0(y)$.

6. The full line containing an edge e of $\mathcal{P}_0(y)$ is denoted by $\ell(e)$.

3 Original definition and the main theorem

The polyhedron $\mathcal{P}(y)$ is called *generic* if it has the following properties.

(i) (i-a) Each edge e of $\mathcal{P}_0(y)$ for which $\ell(e)$ does not end at a parabolic fixed point has an edge cycle of length three.

(i-b) If $\ell(e)$ ends at a parabolic fixed point ζ, then
- e has an edge cycle of length three or four,
- and every transformation entering into the cycle fixes ζ.

(ii)(ii-a) Three edges emanate from each vertex v of $\mathcal{P}_0(y)$.

(ii-b) For at most one of them e, $\ell(e)$ ends at a parabolic fixed point ζ.

(ii-c) The order of v is either four or five.
- In the latter case, three of the four transformations $\neq$ id. associated with v are parabolic and fix the end point ζ of $\ell(e)$ for an edge e emanating from v.

(iii)(iii-a) Every boundary vertex v^* is an end point of exactly one edge e of $\mathcal{P}_0(y)$.

(iii-b) The vertex cycle at v^* has length three or four.
- In the latter case,

(iii-b1) either all the transformations are parabolic and fix the other end of e,

(iii-b2) or they all lie in a cyclic loxodromic subgroup and $y \in \partial\mathbb{H}^3$.

(iv)(iv-a) No edges of $\mathcal{P}_0(y)$ end at a rank one cusp ζ of $\mathcal{P}(y)$
but two faces of $\mathcal{P}(y)$ are tangent to ζ with a face pairing transformations that fixes ζ.

(iv-b) Every rank two cusp ζ is the end point of four or six edges of $\mathcal{P}_0(y)$.

Theorem 2. *Let G be a Kleinian group without elliptic elements. There are dense sets of points $\mathcal{G}^* \subset \partial\mathbb{H}^3, \mathcal{G} \subset \mathbb{H}^3$ such that for any $y \in \mathcal{G}$, or for any $y \in \mathcal{G}^* \cap \Omega(G)$, $\mathcal{P}(y)$ is a generic fundamental polyhedron for G.*

4 Problems

1. There is a contradiction between (ii-a) and (ii-c).

   Suppose there is a vertex v with its order five. Then, by calculating the Euler characteristic, you can see that the neighborhoods of v is constructed by four tetrahedra and one square pyramid. Since each of them are images of neighborhoods of vertices of $\mathcal{P}_0(y)$, there is a vertex from which four edges emanate. This contradicts (ii-a).

   On the other hand, if we assume (ii-a), the discussion above says that there is no vertex in the interior of a length four edge. But I could not find the proof in [JM88] which guaranteed this result, and I cannot prove it by myself now.

2. It seems that there is also a contradiction between (i) and (iii-b2).

   Suppose that there is a boundary vertex v^* satisfying (iii-b2). Then, by (iii-a), the length of the cycle of the edge, say e, ending at v^* is four. Then, by (i-a), the other end point of e must be a parabolic fixed point. In this case, by (i-b), the transformations constructing e must be parabolic, a contradiction.

   So this argument says that there is no boundary vertex satisfying (iii-b2), but this part of the proof in [JM88, Theorem 4.6] is owned by [JM88, Lemma 4.4], and it says there is a possibility of existence of boundary vertices satisfying (iii-b2).

References

[JM88] T. Jørgensen and A. Marden, *Generic fundamental polyhedra for Kleinian groups*, Holomorphic functions and moduli, Vol. II (Berkeley, CA, 1986), Math. Sci. Res. Inst. Publ., vol. 11, Springer, New York, 1988, pp. 69–85.