A note on generic fundamental polyhedra

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1 Preliminaries

Let $\mathbb{H}^3 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_3 > 0\}$ be the upper half-space model of hyperbolic space. We regard its boundary in $\mathbb{R}^3 \cup \{\infty\}$ as $\mathbb{C} \cup \{\infty\}$, which is called the sphere at infinity, and we denote it by $\partial \mathbb{H}^3$. Let $\mathbb{B}^3$ be the open unit sphere centred at the origin of $\mathbb{R}^3$, and we regard it as the Poincaré ball model of hyperbolic space; indeed, an isometry, say $\Phi$, from $\mathbb{H}^3$ to $\mathbb{B}^3$ is given by the reflection along $\mathbb{C}$, followed by the inversion along the sphere of radius $\sqrt{2}$ and centre $(0, 0, 1)$. Then $\Phi(\partial \mathbb{H}^3) = \partial \mathbb{B}^3$ and $\Phi|_\mathbb{C}$ coincide with the stereographic projection from $(0, 0, 1)$. For a hyperbolic plane $\mathbb{P}$ in $\mathbb{H}^3$, we denote its closure in $\mathbb{H}^3 := \mathbb{H}^3 \cup \partial \mathbb{H}^3$ by $\bar{\mathbb{P}}$, and its boundary in $\mathbb{H}^3$ (i.e., $\bar{\mathbb{P}} \cap \partial \mathbb{H}^3$) by $\partial \mathbb{P}$. Any hyperbolic plane in $\mathbb{B}^3$ is the image of some hyperbolic plane $\mathbb{P}$ in $\mathbb{H}^3$ by $\Phi$, and it is the intersection with $\mathbb{B}^3$ of a Euclidean sphere or plane, say $\mathbb{P}^\ast$, orthogonal to $\partial \mathbb{B}^3$. If $\mathbb{P}^\ast$ is a sphere, then we say its centre (in the Euclidean sense) the centre of $\mathbb{P}$ (or $\Phi(\mathbb{P})$), and if $\mathbb{P}^\ast$ is a plane, then the centre of $\mathbb{P}$ (or $\Phi(\mathbb{P})$) means $\infty$. Every point in $\mathbb{R}^3 - \mathbb{B}^3$ uniquely determines a hyperbolic plane in $\mathbb{B}^3$.

Definition. 1. For a point $x \in \mathbb{H}^3$ and a loxodromic or a parabolic element $T \in \text{PSL}_2(\mathbb{C})$, we define $B(x; T) \subset \mathbb{H}^3$ as follows:

(a) If $x \in \mathbb{H}^3$ with $T(x) \neq x$, then $B(x; T)$ is defined as the bisecting perpendicular geodesic plane of the geodesic segment joining $x$ and $T^{-1}(x)$.

(b) If $x \in \partial \mathbb{H}^3$ with $T(x) \neq x$, then $B(x; T)$ is a geodesic plane defined (and well-defined) as a limit of $B(x'; T)$ by a convergence of $x' \in \mathbb{H}^3$ to $x$.

(c) If $x$ is a fixed point of $T$, then $B(x; T) := \{x\}$.

2. For a point $x \in \mathbb{H}^3$, we denote the centre of $\Phi(B(x; T))$ by $C(x; T)$.

The following lemma says that the definition of $B(x; T)$ for a fixed point $x$ is well-defined.

Lemma 1 ([JM88, Lemma 2.4]). Let $T \in \text{PSL}_2(\mathbb{C})$, and let $x \in \partial \mathbb{H}^3$ with $T(x) \neq x$. 

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1. If $T$ is loxodromic, then $\partial B(x; T)$ does not pass through any fixed point of $T$. As $x$ approaches a fixed point of $T$, then $B(x; T)$ converges to the point.

2. If $T$ is parabolic, then $\partial B(x; T)$ passes through the fixed point of $T$. There is a suitable approach of $x$ to the fixed point such that $B(x; T)$ converges to the point.

When $x \in \mathbb{H}^3$, the actual coordinate of $C(x; T)$ is obtained by the following calculations:

$$a := -\Phi(x) + \left(1 - |\Phi(x)|^2\right) \left(\Phi(T^{-1}x) - \Phi(x)\right)^*,$$

$$b := a^* + \left(1 - |a|^2\right) (\Phi(x) + a)^*,$$

$$C(x; T) = \left\{ \Phi(x) + \left(1 - |\Phi(x)|^2\right) (\Phi(x) + b)^* \right\}^*,$$

where $y^*$ means the complex conjugate of $y^{-1}$. You can see that $C(x; T)$ converges to the expression in [JM88, 2.9] as $x_3 \to 0$, if you calculate $C(x; T)$ explicitly.

2 Notations and terminologies

1. First we define a Dirichlet fundamental polyhedron $\mathcal{P}_0(y)$ as follows:

$$\mathcal{P}_0(y) := \begin{cases} 
\{ x \in \mathbb{H}^3 \mid d(x, y) \leq d(y, Tx), \forall T \in G \} & \text{if } y \in \mathbb{H}^3, \\
\left\{ x \in \mathbb{H}^3 \mid x \text{ lies in the closure of the component of } \mathbb{H}^3 - B(y; T) \text{ that is adjacent to } y, \forall T \in G, T \neq id. \right\} & \text{if } y \in \partial \mathbb{H}^3.
\end{cases}$$

- When $y \in \partial \mathbb{H}^3$, $\mathcal{P}_0(y)$ is well defined and is a fundamental polyhedron only when $y$ has special properties, for example, when $y \in \Omega(G)$, where $\Omega(G)$ is the set of ordinary points with respect to $G$ on $\partial \mathbb{H}^3$.
- The point $y$ is called the center of $\mathcal{P}_0(y)$.
- $\mathcal{P}(y) := \overline{\mathcal{P}_0(y)} \cap (\mathbb{H}^3 \cup \Omega(G))$.

2. Associate with each edge $e$ of $\mathcal{P}_0(y)$, is an edge cycle of length $k$, $(T_1 = id., T_2, \ldots, T_k, T_{k+1} = T_1 = id.)$, where $(\mathcal{P}(y), T_2\mathcal{P}(y), \ldots, T_k\mathcal{P}(y))$ is the cyclic arrangement of polyhedra about $e$ in the $G$-orbit of $\mathcal{P}(y)$.

Equivalently, $k$ is the number of disjoint edges of $\mathcal{P}_0(y)$ that are equivalent to $e$ under $G$.

3. The order $k$ of a vertex $v$ of $\mathcal{P}_0(y)$ is the number of distinct vertices of $\mathcal{P}_0(y)$ that are equivalent to $v$ under $G$. 


Equivalently, $k$ is the number of polyhedra, 
$\mathcal{P}_0(y), T_2 \mathcal{P}_0(y), \ldots, T_k \mathcal{P}_0(y)$ 
in the $G$-orbit of $\mathcal{P}(y)$ that share the vertex $v$.

- The transformations $T_i \in G$, are said to be associated with $v$.

4. A cusp of $\mathcal{P}(y)$ is a parabolic fixed point that lies in the Euclidean closure of $\mathcal{P}(y)$.
- It is of rank one or rank two according to the rank of the parabolic subgroup that fixes it.

5. Boundary vertices and boundary edges of $\mathcal{P}(y)$ are those lie in $\Omega(G)$.
- Associated with each boundary vertex is a vertex cycle analogous to the edge cycles of $\mathcal{P}_0(y)$.

6. The full line containing an edge $e$ of $\mathcal{P}_0(y)$ is denoted by $\ell(e)$.

3 Original definition and the main theorem

The polyhedron $\mathcal{P}(y)$ is called generic if it has the following properties.

(i) (i-a) Each edge $e$ of $\mathcal{P}_0(y)$ for which $\ell(e)$ does not end at a parabolic fixed point has an edge cycle of length three.

(i-b) If $\ell(e)$ ends at a parabolic fixed point $\zeta$, then
- $e$ has an edge cycle of length three or four,
- and every transformation entering into the cycle fixes $\zeta$.

(ii)(ii-a) Three edges emanate from each vertex $v$ of $\mathcal{P}_0(y)$.

(ii-b) For at most one of them $e$, $\ell(e)$ ends at a parabolic fixed point $\zeta$.

(ii-c) The order of $v$ is either four or five.
- In the latter case, three of the four transformations $\neq \text{id}$ associated with $v$ are parabolic and fix the end point $\zeta$ of $\ell(e)$ for an edge $e$ emanating from $v$.

(iii)(iii-a) Every boundary vertex $v^*$ is an end point of exactly one edge $e$ of $\mathcal{P}_0(y)$.

(iii-b) The vertex cycle at $v^*$ has length three or four.
- In the latter case,
  (iii-b1) either all the transformations are parabolic and fix the other end of $e$,
  (iii-b2) or they all lie in a cyclic loxodromic subgroup and $y \in \partial \mathbb{H}^3$.

(iv)(iv-a) No edges of $\mathcal{P}_0(y)$ end at a rank one cusp $\zeta$ of $\mathcal{P}(y)$ but two faces of $\mathcal{P}(y)$ are tangent to $\zeta$ with a face pairing transformations that fixes $\zeta$. 

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(iv-b) Every rank two cusp $\zeta$ is the end point of four or six edges of $\mathcal{P}_0(y)$.

**Theorem 2.** Let $G$ be a Kleinian group without elliptic elements. There are dense sets of points $G^* \subset \partial \mathbb{H}^3, \mathcal{G} \subset \mathbb{H}^3$ such that for any $y \in \mathcal{G}$, or for any $y \in G^* \cap \Omega(G)$, $\mathcal{P}(y)$ is a generic fundamental polyhedron for $G$.

4 Problems

1. There is a contradiction between (ii-a) and (ii-c).

   Suppose there is a vertex $v$ with its order five. Then, by calculating the Euler characteristic, you can see that the neighborhoods of $v$ is constructed by four tetrahedra and one square pyramid. Since each of them are images of neighborhoods of vertices of $\mathcal{P}_0(y)$, there is a vertex from which four edges emanate. This contradicts (ii-a).

   On the other hand, if we assume (ii-a), the discussion above says that there is no vertex in the interior of a length four edge. But I could not find the proof in [JM88] which guaranteed this result, and I cannot prove it by myself now.

2. It seems that there is also a contradiction between (i) and (iii-b2).

   Suppose that there is a boundary vertex $v^*$ satisfying (iii-b2). Then, by (iii-a), the length of the cycle of the edge, say $e$, ending at $v^*$ is four. Then, by (i-a), the other end point of $e$ must be a parabolic fixed point. In this case, by (i-b), the transformations constructing $e$ must be parabolic, a contradiction.

   So this argument says that there is no boundary vertex satisfying (iii-b2), but this part of the proof in [JM88, Theorem 4.6] is owned by [JM88, Lemma 4.4], and it says there is a possibility of existence of boundary vertices satisfying (iii-b2).

References